Opakování: Implicitní funkce

$$F(x,y)=0 \quad /d$$

$$dF(x,y)=d0$$

$$\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy=0$$

Chceme $\frac{dy}{dx}$ čili „vydělím" dx

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

$$\boxed{\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}}$$

Př: $y-\frac{1}{2}\sin y=x \qquad y'(x)=?$

Řeš. $\frac{dy}{dx}=-\frac{-1 \cdot 2}{1-\frac{1}{2}\cos y \cdot 2}=\frac{2}{2-\cos y}$

$$\frac{dy}{dx}(0)=? \quad =\frac{2}{2-\cos(y(0))}=\frac{2}{2-1}=\underline{\underline{2}}$$

$y(0)$: $y-\frac{1}{2}\sin y=0 \Rightarrow y=0$

Vícerozměrný případ:

Př: $F(x,y,z) \equiv x^2+y^2+z^2-3xyz=0$

Rovnice definuje $z(x,y)$

Najděte $\frac{\partial z}{\partial x}$ a $\frac{\partial z}{\partial y}$.

(explicitně $z = \operatorname{arctg}(x+y)$

Řešení: $\frac{\partial z}{\partial x} \left(= \frac{dz}{dx}\right)$

$$F(x,y,z)=0 \quad /d$$

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = 0 \quad /:dx$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\underbrace{\frac{dy}{dx}}_{=0} + \frac{\partial F}{\partial z}\frac{dz}{dx} = 0$$

$$\frac{dz}{dx} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{dz}{dy} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

konkrétně: $\frac{dz}{dx} = -\frac{2x-3yz}{2z-3xy}$

$\frac{dz}{dx}(1,1)$ to je problém, protože z není jednoznačné

$z(1,1)$: $x^2+y^2+z^2-3xyz=0$

$x=1$ $y=1$ $\rightarrow$ $2+z^2-3z=0$ lok. funkce

$\Rightarrow z = \left\langle \begin{matrix}1\\2\end{matrix}\right.$

graf z:

Zvolím $z = 1$ (a ne 2)(třeba)

$$\left.\frac{dz}{dx}\right|_{(1,1,1)} = -\left.\frac{2x-3yz}{2z-3xy}\right|_{(1,1,1)} = -\frac{2\cdot 1-3\cdot 1\cdot 1}{2\cdot 1-3\cdot 1\cdot 1} = \underline{\underline{-1}}$$

V 2. případě $z=2$

$$\left.\frac{dz}{dx}\right|_{(1,1,2)} = -\frac{2\cdot 1-3\cdot 1\cdot 2}{2\cdot 2-3\cdot 1\cdot 1} = \underline{\underline{4}}$$

<u>Př:</u>

$$\begin{aligned} x^2+y^2+z^2-3xyz &= 0 \\ x-y^2z &= 0 \end{aligned} \quad (*)$$

Tato soustava definuje $(y,z)=f$, $f:\mathbb{R}\to\mathbb{R}^2$ jako funkci x.

Otázka: $df(x) = ?$

$$F:\mathbb{R}^3\to\mathbb{R}^2 \quad F(x,y,z) = \begin{pmatrix} x^2+y^2+z^2-3xyz \\ x-y^2z \end{pmatrix}$$

$$*:\quad F(x,y,z)=0 \quad /d$$

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial(y,z)}\cdot df = 0 \quad /:dx$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial(y,z)}\frac{df}{dx} = 0 \quad \Big/\left(\frac{\partial F}{\partial(y,z)}\right)^{-1}\cdot$$

$$\frac{df}{dx} = -\left(\frac{\partial F}{\partial(y,z)}\right)^{-1}\cdot\frac{\partial F}{\partial x}$$

$$\frac{\partial F}{\partial(y,z)} = \begin{array}{c} \begin{array}{cc} y & z \end{array} \\ \begin{pmatrix} 2y-3xz & 2z-3xy \\ -2yz & -y^2 \end{pmatrix} \end{array}$$

$$\frac{\partial F}{\partial x} = \begin{pmatrix} 2x-3yz \\ 1 \end{pmatrix}$$

$$\left.\frac{df}{dx}\right|_{(1,1,1)} = -\left.\left(\frac{\partial F}{\partial(y,z)}\right)^{-1}\right|_{(1,1,1)} \cdot \left.\frac{\partial F}{\partial x}\right|_{(1,1,1)}$$

Speciální případ

$$= -\begin{pmatrix}-1 & -1\\ -2 & -1\end{pmatrix}^{-1}\cdot\begin{pmatrix}-1\\ 1\end{pmatrix}$$

$$= -\begin{pmatrix}1 & -1\\ -2 & 1\end{pmatrix}\cdot\begin{pmatrix}-1\\ 1\end{pmatrix} = \underline{\underline{\begin{pmatrix}2\\ -3\end{pmatrix}}}$$

Obecné: $F: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$

$F(x,y) = 0$ $\quad x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

m rovnic

a $\det\left(\frac{\partial F}{\partial y}\right) \neq 0$

potom
$$\boxed{\underbrace{\frac{dy}{dx}}_{m\times n} = -\underbrace{\left(\frac{\partial F}{\partial y}\right)^{-1}}_{m\times m} \cdot \underbrace{\frac{\partial F}{\partial x}}_{m\times n}}$$

Inverzní zobrazení

jednorozměrný případ:

$$f^{-1}(f(x)) = x \quad / \frac{d}{dx}$$

$$f^{-1\prime}(f(x)) \cdot f'(x) = 1$$

$$f^{-1\prime}(f(x)) = \frac{1}{f'(x)}$$

označím $f(x) = y$

$$\boxed{f^{-1\prime}(y) = \frac{1}{f'(x)}}$$

Př: $\arcsin'(x) = ?$

$$\arcsin(\sin(x)) = \sin(\arcsin(x)) = x$$

$$\arcsin'(y) = \frac{1}{\sin'(x)} = \frac{1}{\cos x}$$

$y = \sin x$

$$\cos x = \sqrt{1-\sin^2 x}$$

$$\underline{\arcsin'(y) = \frac{1}{\sqrt{1-y^2}}}$$

Vícerozměrný případ

Věta: Nechť $F: \mathbb{R}^n \to \mathbb{R}^n$ spojitě diferencovatelná v $x_0 \in \mathbb{R}^n$, $F(x_0) = y_0$ a matice parc. derivací je regulární, tj. $\left|\frac{\partial F^i}{\partial x_j}\right| \neq 0$ potom $\exists O(x_0)$ a $\exists O(y_0)$ takové, že F je bijekce z $O(x_0)$ na $O(y_0)$, F^{-1} je diferenc. v y_0 a platí:

$$\frac{\partial F^{-1\,i}}{\partial x_j}(y_0) = \left(\frac{\partial F^i}{\partial x_j}(x_0)\right)^{-1}.$$