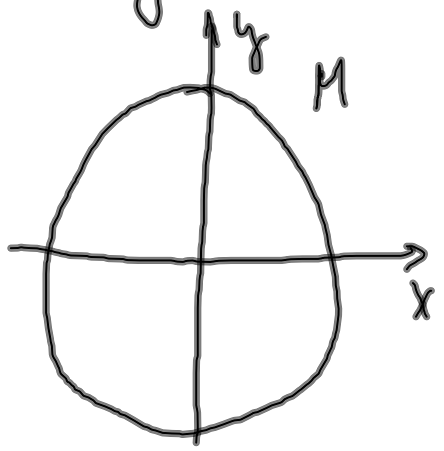
Věta o implicitním a inverzním zobrazení

Příklad: $x^2+y^2=1$



$F(x,y)=0$

$F(x,y)=x^2+y^2-1$

Věta (o implicitní funkci v $\mathbb{R}^2$)

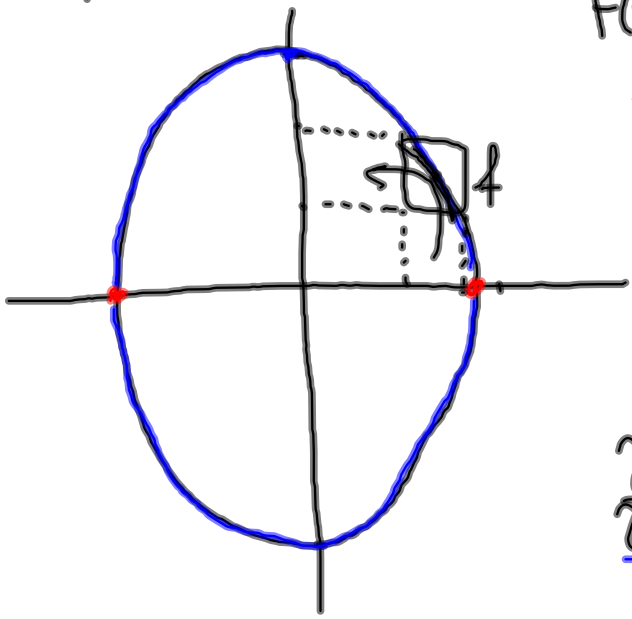
Nechť $F: M\subset\mathbb{R}^2 \to \mathbb{R}$ spojitě diferencovatelná, $(x_0,y_0)\in M$
v (x_0,y_0) $F(x_0,y_0)=0$ a $\frac{\partial F}{\partial y}(x_0,y_0)\neq 0$

Potom $\exists$ okolí y_0 $I=(y_0-\delta, y_0+\delta)$, okolí x_0 $J=(x_0-\varepsilon, x_0+\varepsilon)$
na němž $\forall x\in(x_0-\varepsilon, x_0+\varepsilon)$ $\exists!\, y\in(y_0-\delta, y_0+\delta)$ splňující
$F(x,y)=0$

Jinými slovy $\exists$ okolí (x_0,y_0) $I\times J$ na němž rovnice $F(x,y)=0$
implicitně definuje y jako funkci x; $f: I\to J$ $x\mapsto y$
f - je spojitá, diferencovatelná a platí

$$f'(x_0)=-\frac{\frac{\partial F}{\partial x}(x_0,y_0)}{\frac{\partial F}{\partial y}(x_0,y_0)}$$

Zpět k příkladu



$F(x,y) = x^2 + y^2 - 1 = 0$

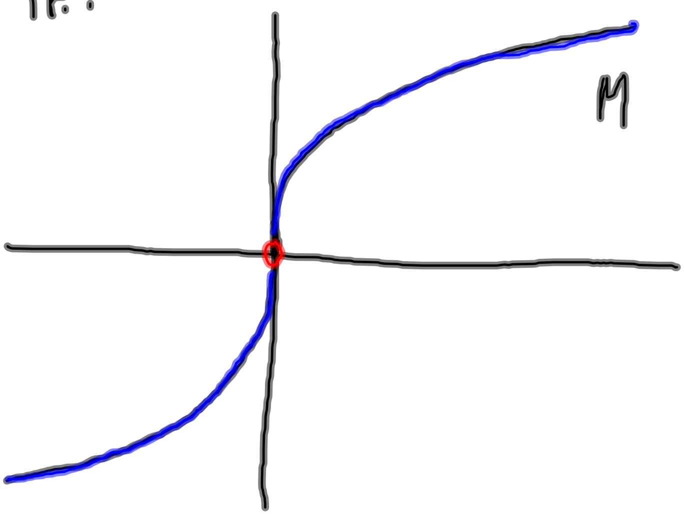
- spoj. difer.

$y = f(x)$

$\frac{\partial F}{\partial y}(x_0,y_0) \neq 0$

$\frac{\partial F}{\partial y}(x_0,y_0) = 2y_0 \neq 0$

Př.:



$F(x,y) = x - y^3 = 0$

$y = f(x)$

$\frac{\partial F}{\partial y}(x_0,y_0) = -3y_0^2 \neq 0$

$y = \sqrt[3]{x} = f(x)$

Věta (o implicitní funkci v $\mathbb{R}^3$)

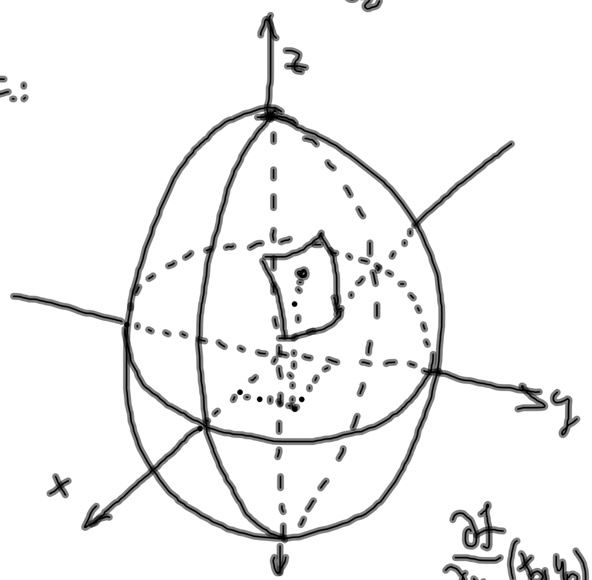
Buď $F: M \subset \mathbb{R}^3 \to \mathbb{R}$ - spoj. difer. na M, $(x_0, y_0, z_0) \in M$

$F(x_0, y_0, z_0) = 0$ a $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$.

Pak $\exists$ okolí (x_0, y_0).. I, okolí z_0... J : $\forall (x, y) \in I$ $\exists! z \in J$ splňují $F(x, y, z) = 0$, tedy definuje z jako funkci x, y $\exists f: I \to J$ $F(x, y, f(x,y)) = 0$. f - spoj. diferencov.

$$\frac{\partial f}{\partial x}(x_0, y_0) = -\frac{\frac{\partial F}{\partial x}(x_0, y_0, z_0)}{\frac{\partial F}{\partial z}(x_0, y_0, z_0)}, \quad \frac{\partial f}{\partial y}(x_0, y_0) = -\frac{\frac{\partial F}{\partial y}(x_0, y_0, z_0)}{\frac{\partial F}{\partial z}(x_0, y_0, z_0)}.$$

Př.:



$F(x, y, z) = x^2 + y^2 + z^2 - 1$

$(x_0, y_0, z_0) = \left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$

$\frac{\partial F}{\partial z}(x_0, y_0, z_0) = 2z_0 \neq 0$

$$\frac{\partial f}{\partial x}(x_0, y_0) = -\frac{\frac{\partial F}{\partial x}(x_0, y_0, z_0)}{\frac{\partial F}{\partial z}(x_0, y_0, z_0)} = \frac{-2x_0}{2z_0}$$

$f(x, y) = \sqrt{1 - x^2 - y^2}$

$$\frac{\partial f}{\partial x}\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{2\,\frac{1}{2}}{2\,\frac{\sqrt{2}}{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\frac{\partial f}{\partial x}(x, y) = \frac{1}{2}\frac{-2x}{\sqrt{1 - x^2 - y^2}} =$$

$$= -\frac{x}{\sqrt{1 - x^2 - y^2}}$$

$$\frac{\partial f}{\partial x}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{-\frac{1}{2}}{\sqrt{1 - \frac{1}{4} - \frac{1}{4}}} = -\frac{\frac{1}{2}}{\frac{1}{\sqrt{2}}} = -\frac{\sqrt{2}}{2}$$

Věta (o implicitním zobrazení obecně)

Buď $F: M \subset \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ – spoj. diferencov. $x_0 \in \mathbb{R}^m$, $y_0 \in \mathbb{R}^n$

$F(x_0, y_0) = 0 \qquad \det\left(\frac{\partial F^i}{\partial y^j}\right) \neq 0$

Potom $\exists$ okolí $O(x_0)$, okolí $O(y_0)$: $\forall x \in O(x_0)\ \exists!\, y \in O(y_0)$, $F(x,y) = 0$

$\exists f: O(x_0) \to O(y_0) \quad y = f(x)$ – spoj. difer.

$$\left(\frac{\partial f}{\partial x}\right) = -\left(\frac{\partial F}{\partial y}\right)^{-1} \circ \left(\frac{\partial F}{\partial x}\right).$$

Příklad $\qquad F: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x,y,z,u) = \begin{pmatrix} x+2y-z+u \\ -2x+y+2z+2u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (x_0,y_0,z_0,u_0) = (1,4,4,-5)$$

$\boxed{z, u = f(x,y)}$

$$\left(\frac{\partial F}{\partial zu}\right) = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix} \qquad \left(\frac{\partial F}{\partial zu}\right)^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$\det(\quad) = -4 \qquad \left(\frac{\partial F}{\partial xy}\right) = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$