Ještě jednou zpět k derivaci složené funkce

Věta: $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$, $g: V \subset \mathbb{R}^m \to \mathbb{R}^k$, $x \in U$, $y = f(x) \in V$

f, g spojitě diferencovatelné na U, V

Potom: $g \circ f$ je diferencovatelná v x

$$(g \circ f)'(x) = g'(f(x)) \circ f'(x)$$

$$\begin{pmatrix} \frac{\partial g\circ f^1}{\partial x^1}(x) & \cdots & \frac{\partial g\circ f^1}{\partial x^n}(x) \\ \vdots & & \\ \frac{\partial g\circ f^k}{\partial x^1} & \cdots & \frac{\partial g\circ f^k}{\partial x^n}(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial g^1}{\partial x^1}(f(x)) & \cdots & \frac{\partial g^1}{\partial x^m}(f(x)) \\ \vdots & & \\ \frac{\partial g^k}{\partial x^1}(f(x)) & \cdots & \frac{\partial g^k}{\partial x^m}(f(x)) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(x) & \cdots & \frac{\partial f^1}{\partial x^n}(x) \\ \vdots & & \\ \frac{\partial f^m}{\partial x^1}(x) & \cdots & \frac{\partial f^m}{\partial x^n}(x) \end{pmatrix}$$

Například

$$\frac{\partial g\circ f^1}{\partial x^1}(x) = \frac{\partial g^1}{\partial x^1}(f(x)) \cdot \frac{\partial f^1}{\partial x^1}(x) + \frac{\partial g^1}{\partial x^2}(f(x)) \cdot \frac{\partial f^2}{\partial x^1}(x) + \dots + \frac{\partial g^1}{\partial x^m}(f(x)) \cdot \frac{\partial f^m}{\partial x^1}(x)$$

$$a^1_1 \cdot b^1_1 + a^1_2 b^2_1 + a^1_3 b^3_1 + \dots + a^1_m b^m_1$$

$$\frac{\partial g\circ f^3}{\partial x^4}(x) = \frac{\partial g^3}{\partial x^1}(f(x)) \cdot \frac{\partial f^1}{\partial x^4} + \frac{\partial g^3}{\partial x^2}(f(x)) \cdot \frac{\partial f^2}{\partial x^4}(x) + \dots + \frac{\partial g^3}{\partial x^m}(f(x)) \cdot \frac{\partial f^m}{\partial x^4}(x).$$

$$g\circ f = \mathrm{id} \qquad (x,y) \longmapsto (x,y)$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (g\circ f)'(x,y)$$

$$(g\circ f)^2(x,y) = y$$

$$(g\circ f)^1(x,y) = x$$

$$f(x,y) = \begin{pmatrix} \sqrt{x^2+y^2} \\ \operatorname{arctg} \frac{y}{x} \end{pmatrix} \qquad g(x,y) = \begin{pmatrix} x\cos y \\ x \sin y \end{pmatrix}$$

$$\cos \operatorname{arctg} t = \frac{1}{\sqrt{1+t^2}} \qquad \sin \operatorname{arctg} t = \frac{t}{\sqrt{1+t^2}}$$

$$\frac{\partial (g\circ f)^1}{\partial x}(x,y) = \frac{\partial g^1}{\partial x}(f(x,y)) \cdot \frac{\partial f^1}{\partial x}(x,y) + \frac{\partial g^1}{\partial y}(f(x,y)) \cdot \frac{\partial f^2}{\partial x}(x,y)$$

$$\frac{\partial (g\circ f)^1}{\partial x}(x,y) = \cos\left(\operatorname{arctg}\frac{y}{x}\right) \cdot \frac{x}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}\,\sin\left(\operatorname{arctg}\frac{y}{x}\right) \cdot \frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}} =$$

$$= \frac{1}{\sqrt{1+\frac{y^2}{x^2}}} \cdot \frac{x}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}\,\frac{\frac{y}{x}}{\sqrt{1+\frac{y^2}{x^2}}} \cdot \frac{-y}{x^2+y^2} =$$

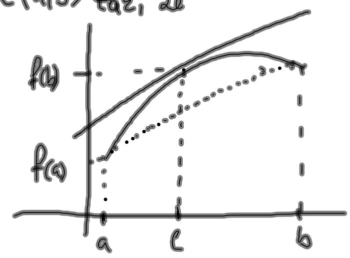
$$= \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{x}{\sqrt{x^2+y^2}} + \sqrt{x^2+y^2}\,\frac{y}{\sqrt{x^2+y^2}} \cdot \frac{y}{x^2+y^2} =$$

$$= \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} ,= 1$$

Zopakování věty o střední hodnotě $\mathbb{R}\to\mathbb{R}$

Věta (Lagrangeova věta o střední hodnotě)
Buď $f:[a,b]\to\mathbb{R}$ spojitá a má derivaci na (a,b)
potom existuje $c\in(a,b)$ tak, že

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



Příklad: $[5,10]$, $\ln$

$$\underbrace{\ln(10)-\ln(5)}_{\ln 2}=\underbrace{\ln'(c)}_{=\frac{1}{c}}\cdot\underbrace{(10-5)}_{5}$$

$$\ln 2=\frac{5}{c}$$

$$1=\frac{5}{5}<\frac{c}{5}<\frac{10}{5}=2$$

$$1>\frac{5}{c}>\frac{1}{2}$$

$$1>\ln 2>\frac{1}{2}$$

$$e^1>e^{\ln 2}$$

$$e>2$$

$$e^{\ln 2}>e^{\frac{1}{2}}$$

$$2>\sqrt{e}$$

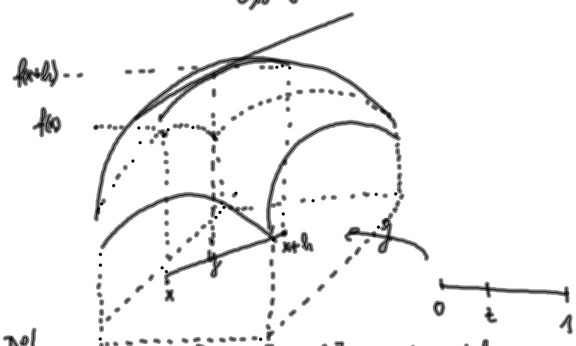
$$4>e$$

$$2<e<4$$

Věta (o střední hodnotě funkce více proměnných)
Buď $f:U\subset\mathbb{R}^n\to\mathbb{R}$, $x\in U$ taková, že obsahuje úsečku $[x,x+h]$, $h\in\mathbb{R}^n$, f má v každém bodě úsečky $[x,x+h]$ derivaci ve směru h.

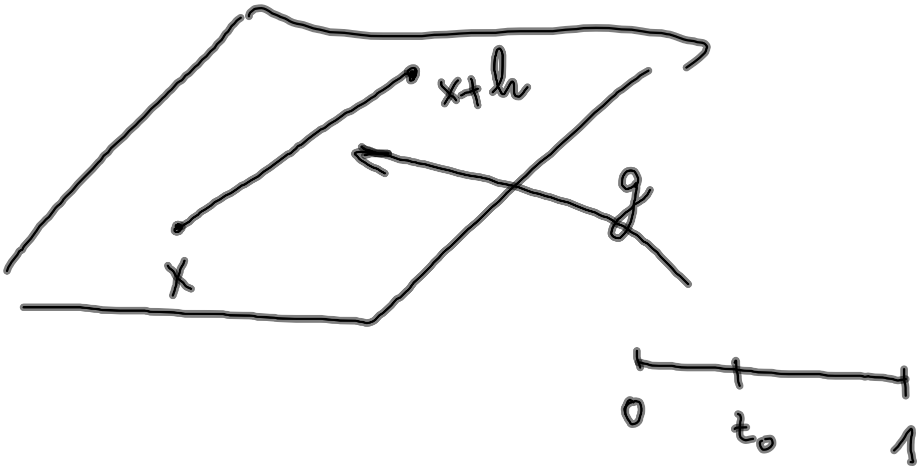
Potom existuje bod $y\in[x,x+h]$ takový, že

$$f(x+h)-f(x)=\frac{\partial f}{\partial h}(y).$$



Důkaz: $g:[0,1]\to[x,x+h]$ $g(t)=x+t\cdot h$

$f\circ g:[0,1]\to\mathbb{R}$ Použijeme na ni Lagrangeovu větu funkce jedné proměn.



Důkaz: $g:[0,1]\longrightarrow[x,x+h]$ $\quad g(t)=x+t\cdot h$

$f\circ g:[0,1]\longrightarrow\mathbb{R}$ Použijme na ni Lagrangeovu větu funkce jedné proměn.

$$f\circ g(1)-f\circ g(0)=\underbrace{(f\circ g)'(t_0)}$$

$$f(g(1))-f(g(0))$$

$$f(x+h)-f(x) = \frac{\partial f}{\partial h}(\underbrace{g(t_0)}_{=y=x+t_0 h})=\frac{\partial f}{\partial h}(y)$$

Proč to neplatí pro funkce $\longrightarrow\mathbb{R}^m$

Př.: $f:\mathbb{R}\longrightarrow\mathbb{R}^2$ $\quad f(t)=(\cos t,\sin t)$

$[0,2\pi]$

$f(0)=f(2\pi]$ $\quad f(2\pi)-f(0)=0$ ale derivace $\frac{\partial f}{\partial h}\neq 0$