$f: \mathbb{R}^n \to \mathbb{R}^m \quad x \in \mathbb{R}^n \quad l: \mathbb{R}^n \to \mathbb{R}^m$

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - l(h) \|}{\| h \|} = 0 \qquad l \ldots f'(x) \ldots df(x), Df(x)$$

Veta: $f: \mathbb{R}^n \to \mathbb{R}^m$, $f = (f^1, f^2, \ldots, f^m)$, $f^i: \mathbb{R}^n \to \mathbb{R}$ jsou diferencovatelné potom je diferencovatelná f

$$f'(x) = (f^{1\prime}(x), f^{2\prime}(x), \ldots, f^{m\prime}(x)).$$

Příklad

$f(x,y) = 2x + 3y \quad (x_0, y_0) = (1,2) \quad l(h_1, h_2) = 2h_1 + 3h_2.$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{| f(x_0+h_1, y_0+h_2) - f(x_0,y_0) - l(h_1,h_2) |}{\| (h_1,h_2) \|} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{| 2(x_0+h_1) + 3(y_0+h_2) - (2+6) - (2h_1 + 3h_2) |}{\sqrt{h_1^2 + h_2^2}} =$$

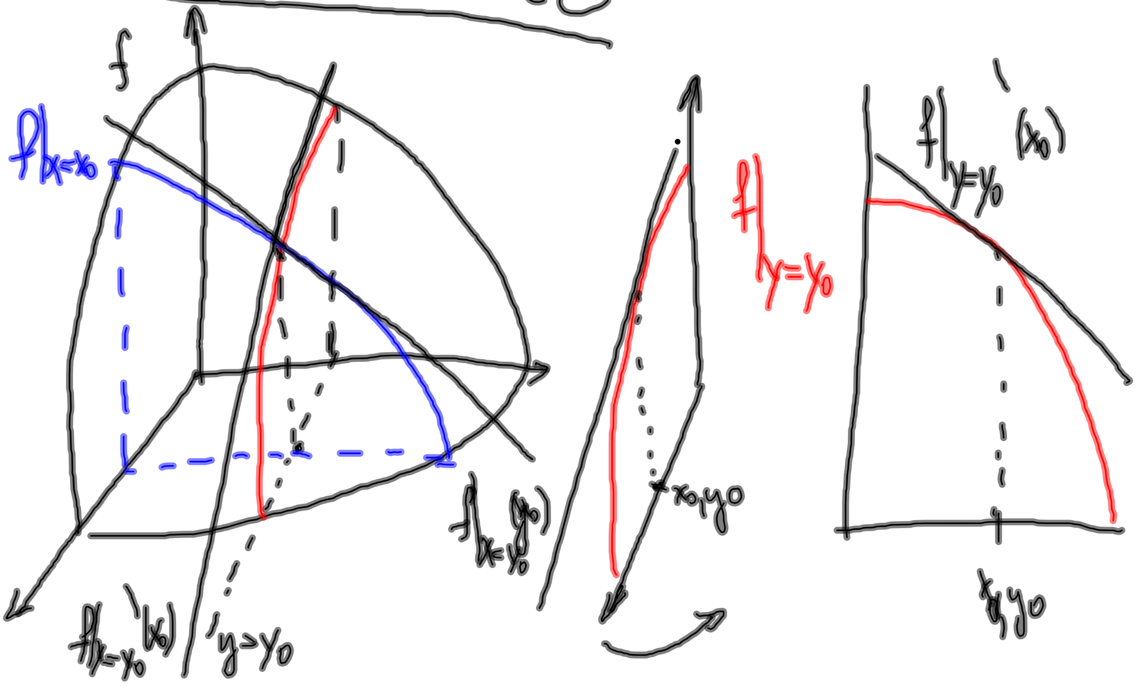
$$= \lim_{(h_1,h_2)\to(0,0)} \frac{| \cancel{2} + \cancel{2h_1} + \cancel{6} + \cancel{3h_2} - \cancel{8} - \cancel{2h_1} - \cancel{3h_2} |}{\sqrt{h_1^2 + h_2^2}} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{0}{\sqrt{h_1^2 + h_2^2}} = 0$$

Značení $\quad \underline{l(h_1, h_2) = 2h_1 + 3h_2} \qquad dx \; (h_1,h_2) \mapsto h_1$

$\underline{l = 2 \cdot dx + 3dy} \qquad dy \; (h_1,h_2) \mapsto h_2$

Parciální derivace



$$\frac{\partial f}{\partial x}(x_0,y_0) = \left(f|_{y=y_0}\right)'(x_0)$$ – parciální derivace podle x.

$$\frac{\partial f}{\partial y}(x_0,y_0) = \left(f|_{x=x_0}\right)'(y_0)$$ – parciální derivace podle y.

Příklad

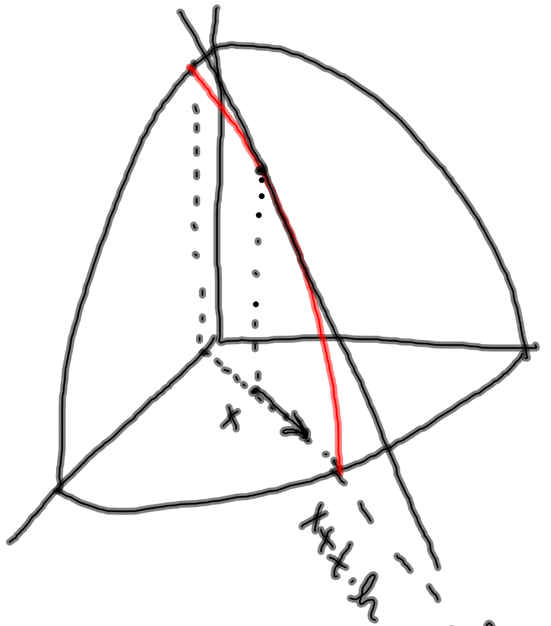
$$f(x,y) = \sin(x)\cdot\cos(y)$$

$$\frac{\partial f}{\partial x}(x_0,y_0) = \left(f\Big|_{y=y_0}\right)' = \left(\sin(x)\cdot\cos(y_0)\right)'\Big|_{x=x_0} =$$

$$= \left(\cos(x)\cdot\cos(y_0)\right)\Big|_{x=x_0} = \cos(x_0)\cdot\cos(y_0).$$

$$\frac{\partial f}{\partial y}(x_0,y_0) = \sin(x_0)\cdot(-\sin(y_0)) = -\sin x_0\cdot\sin y_0$$

Derivace podle vektoru (ve směru)



$$\frac{\partial f}{\partial h}(x) = \lim_{t\to 0} \frac{f(x+t\cdot h)-f(x)}{t}$$

$D_h(x)$

$h=(1,0)$ $\quad \frac{\partial f}{\partial h}(x) = \frac{\partial f}{\partial x}(x)$

$\;\;=(0,1)$ $\quad\quad = \frac{\partial f}{\partial y}(x)$

Příklad

$f(x,y)=2x+3y \qquad (x_0,y_0)=(1,2)$

$h=(2,1)$

$$\frac{\partial f}{\partial h}(x_0,y_0) = \lim_{t\to 0} \frac{f(\overbrace{(x_0,y_0)+t\cdot h}^{(x_0+t\cdot h_1,\, y_0+t\cdot h_2)=(1+t\cdot 2,\, 2+t\cdot 1)}) - f(x_0,y_0)}{t} =$$

$$= \lim_{t\to 0} \frac{2\cdot(1+t\cdot 2)+3(2+t)-8}{t} = \lim_{t\to 0}\frac{2+4t+6+3t-8}{t} =$$

$$= \lim_{t\to 0}\frac{7t}{t} = 7.$$

Spojitá diferencovatelnost

$f: \mathbb{R}^n \to \mathbb{R}$ je spojitě diferencovatelná v x_0 na okolí x_0 $\exists$ všechny parciální derivace a jsou spojité

Věta, (o výpočtu derivace ve směru)

Nechť $f: \mathbb{R}^n \to \mathbb{R}$, $h_1, h_2 \in \mathbb{R}^n$ $\exists \frac{\partial f}{\partial h_1}, \frac{\partial f}{\partial h_2}$ na okolí $x_0 \in \mathbb{R}^n$

potom pro $c_1, c_2 \in \mathbb{R}$ $(c_1, c_2) \neq (0,0)$

$$\frac{\partial f}{\partial (c_1 h_1 + c_2 h_2)}(x_0) = c_1 \frac{\partial f}{\partial h_1}(x_0) + c_2 \frac{\partial f}{\partial h_2}(x_0).$$

<u>Důsledek</u>

Když $f: \mathbb{R}^n \to \mathbb{R}$, $h = (h_1, \dots, h_n) \in \mathbb{R}^n$, f má spojité parciál. derivace na okolí $x_0 \in \mathbb{R}^n$, potom platí

$$\frac{\partial f}{\partial h}(x_0) = h_1 \cdot \frac{\partial f}{\partial x_1}(x_0) + \dots + h_n \frac{\partial f}{\partial x_n}(x_0)$$

Př.: $f(x,y) = 2x + 3y$ $(x_0, y_0) = (1,2)$ $h = (2,1)$

$$\frac{\partial f}{\partial x}(x,y) = 2, \quad \frac{\partial f}{\partial y}(x,y) = 3$$

$$\frac{\partial f}{\partial h}(1,2) = h_1 \cdot \frac{\partial f}{\partial x}(1,2) + h_2 \frac{\partial f}{\partial y}(1,2) =$$

$$= 2 \cdot 2 + 1 \cdot 3 = 7.$$

Věta (o parc. derivacích a diferenciálu)
Nechť $f: \mathbb{R}^n \to \mathbb{R}^m$, $f = (f^1, f^2, \ldots, f^m)$ pokud existují parciální derivace $\frac{\partial f^i}{\partial x_j}(x_0)$ a jsou spojité na okolí x_0.
Potom diferenciál $f'(x_0)$ má matici

$$f'(x_0) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(x_0), & \ldots, & \frac{\partial f^1}{\partial x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x_1}(x_0), & \ldots, & \frac{\partial f^m}{\partial x_n}(x_0) \end{pmatrix}.$$

Zpět k diferenciálu složené funkce.

Věta (o diferenciálu složené funkce podruhé)

$f:\mathbb{R}^n \to \mathbb{R}^m$, $g:\mathbb{R}^m \to \mathbb{R}^l$, spojitě difer. na okolí $x_0, f(x_0)$

potom $g\circ f$ je diferencovatelná a platí:

$$(g\circ f)'(x_0) = g'(f(x_0)) \circ f'(x_0)$$

$$\begin{pmatrix} \frac{\partial (g\circ f)^1}{\partial x_1}(x_0) & \cdots & \frac{\partial (g\circ f)^1}{\partial x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial (g\circ f)^l}{\partial x_1}(x_0) & \cdots & \frac{\partial (g\circ f)^l}{\partial x_n}(x_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial g^1}{\partial x_1}(f(x_0)) & \cdots & \frac{\partial g^1}{\partial x_m}(f(x_0)) \\ \vdots & & \vdots \\ \frac{\partial g^l}{\partial x_1}(f(x_0)) & \cdots & \frac{\partial g^l}{\partial x_m}(f(x_0)) \end{pmatrix} \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(x_0) & \cdots & \frac{\partial f^1}{\partial x_n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x_1}(x_0) & \cdots & \frac{\partial f^m}{\partial x_n}(x_0) \end{pmatrix}$$