Důsledek 8.19 (Newton-Leibnizova formule pro metodu per-partes)

Nechť $u, v : [a,b] \to \mathbb{R}$ spojité derivace

Potom existuje integrál

$$\int_a^b u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx$$

$$= \Big[u(x)v(x)\Big]_a^b - \int_a^b u'(x)v(x)\,dx.$$

$$\int u(x)\, v'(x)\,dx = u(x)\cdot v(x) - \int u'(x)v(x)\,dx$$

$$\int_a^b u(x)\, v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b v'(x)v(x)\,dx$$

Příklad

$$\int_{-1}^{1}(3-4x)e^x\,dx = \begin{vmatrix} u = 3-4x & u' = -4 \\ v' = e^x & v = e^x \end{vmatrix} =$$

$$= \Big[(3-4x)e^x\Big]_{-1}^{1} - \int_{-1}^{1} -4e^x\,dx =$$

$$= (-1e^1 - 7e^{-1}) + 4\big[e^x\big]_{-1}^{1} = -e - 7e^{-1} + 4e - 4e^{-1}$$

$$= 3e - 11e^{-1}$$

Důsledek 8.20 (Newton-Leibnizova formule pro substituci)

Nechť $\varphi : [\alpha, \beta] \to [A, B]$ spojitou derivaci, dále spojitou f na $[A, B]$.

Potom
$$\int_\alpha^\beta f(\varphi(x))\varphi'(x)\,dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx \,.$$

$$\int f(\varphi(x))\varphi'(x)\,dx = G(x) = \int f(x)\,dx$$

$$\int_\alpha^\beta f(\varphi(x))\varphi'(x)\,dx = G(\beta) - G(\alpha) = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

Příklad: $\quad f(x) = \sqrt{x} \quad \varphi(x) = 25 - x^2 \quad \varphi' = \underline{-2x}$

$$\int_3^4 x\boxed{\sqrt{25-x^2}}\,dx = -\frac{1}{2}\int_3^4 \sqrt{25-x^2}\,(-2x)\,dx =$$

$\varphi(3) = 16$

$\varphi(4) = 9$

$$\int_\alpha^\beta f(\varphi(x)) \cdot \varphi'(x)\,dx = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

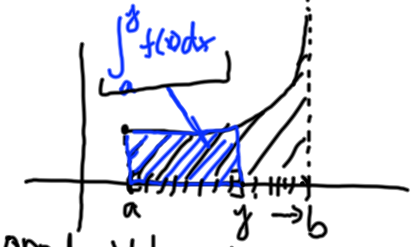
$$= -\frac{1}{2}\int_{16}^{9} \sqrt{x}\,dx = \frac{1}{2}\int_9^{16} \sqrt{x}\,dx = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}x^{3/2}}{3}\right]_9^{16} =$$

$$= \frac{1}{3}\left((\sqrt{16})^3 - (\sqrt{9})^3\right) = \frac{1}{3}(64 - 27) = \frac{37}{3}$$

Title: 4. 24-3:05 odp. (2 of 8)

8.3 Nevlastní Riemannův integrál

f - ne nutně omezená na (a,b) $\left(\begin{matrix} a = -\infty \\ b = \infty \end{matrix}\right)$



- Integrál na $[a,b)$, pokud pro každé $y \in (a,b)$ existuje $\int_a^y f(x)\,dx$ a existuje limita

$$\lim_{y \to b^-} \int_a^y f(x)\,dx = \int_a^b f(x)\,dx$$

- Integrál na $(a,b]$, pokud existuje $y \in (a,b)$ $\int_y^b f(x)\,dx$

$$\lim_{y \to a^+} \int_y^b f(x)\,dx = \int_a^b f(x)\,dx$$

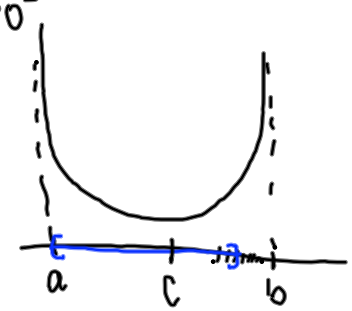
Příklad

$$\int_0^1 \frac{1}{x}\,dx = \lim_{y \to 0^+} \int_y^1 \frac{1}{x}\,dx$$

$\ln x.$

$$= \lim_{y \to 0^+} \left[\ln x\right]_y^1 = \lim_{y \to 0^-} \ln(1) - \ln(y) =$$

$$= \lim_{y \to 0^+} -\ln y = \infty$$



Máme funkci $f:(a,b)\to\mathbb{R}$ nemusí být omezená

$c\in(a,b)$, $\forall y\in(a,b)$ existuje integrál

$$\int_y^c f(x)\,dx \qquad \int_c^y f(x)\,dx$$

Musí existovat limity

$$\lim_{y\to a^+}\int_y^c f(x)\,dx\,,\quad \lim_{y\to b^-}\int_c^y f(x)\,dx$$

Potom definujeme

$$\int_a^b f(x)\,dx = \lim_{y\to a^+}\int_y^c f(x)\,dx + \lim_{y\to b^-}\int_c^y f(x)\,dx$$

Lemma 8.21. Buď $f:(a,b)\to\mathbb{R}$ splňující podmínky z definice nevlastního integrálu na (a,b), $c,d\in(a,b)$, $c<d$.

Potom
$$\int_a^c f(x)\,dx+\int_c^b f(x)\,dx=\int_a^d f(x)\,dx+\int_d^b f(x)\,dx.$$

Důkaz:
$$\int_a^c f(x)\,dx+\int_c^b f(x)\,dx=\lim_{y\to a^+}\int_y^c f(x)\,dx+\lim_{y\to b^-}\int_c^y f(x)\,dx$$
$$=\lim_{y\to a^+}\int_y^c f(x)\,dx+\lim_{y\to b^-}\left(\int_c^d f(x)\,dx+\int_d^y f(x)\,dx\right)=$$
$$=\lim_{y\to a^+}\left(\int_y^c f(x)\,dx+\int_c^d f(x)\,dx\right)+\lim_{y\to b^-}\int_d^y f(x)\,dx=$$
$$=\lim_{y\to a^+}\int_y^d f(x)\,dx+\lim_{y\to b^-}\int_d^y f(x)\,dx=$$
$$=\int_a^d f(x)\,dx+\int_d^b f(x)\,dx$$

Vypočteme

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \underbrace{\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx}_{\frac{\pi}{2}} + \underbrace{\int_{0}^{\infty} \frac{1}{1+x^2}\,dx}_{\frac{\pi}{2}} = \pi$$

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{y\to-\infty} \int_{y}^{0} \frac{dx}{1+x^2} = \lim_{y\to-\infty} \Big[\arctan(x)\Big]_{y}^{0} =$$

$$= \lim_{y\to-\infty} \big(0 - \arctan(y)\big) = -\lim_{y\to-\infty} \arctan(y) = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{y\to\infty} \int_{0}^{y} \frac{dx}{1+x^2} = \lim_{y\to\infty} \Big[\arctan(x)\Big]_{0}^{y} =$$

$$= \lim_{y\to\infty} \big(\arctan(y) - 0\big) = \lim_{y\to\infty} \arctan(y) = \frac{\pi}{2}$$

Príklad

$$\int_0^\infty \frac{1}{x^2}\,dx = \int_0^1 \frac{dx}{x^2} + \int_1^\infty \frac{dx}{x^2} = \infty + 1 = \infty$$

$$\int_0^1 \frac{dx}{x^2} = \lim_{y\to 0^+} \int_y^1 \frac{dx}{x^2} = \lim_{y\to 0^+} \left[-\frac{1}{x}\right]_y^1 =$$

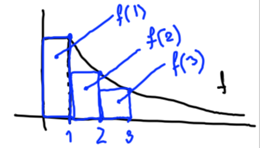
$$= \lim_{y\to 0^+} \left(-1 + \frac{1}{y}\right) = -1 + \lim_{y\to 0^+} \frac{1}{y} = \infty$$

$$\int_1^\infty \frac{dx}{x^2} = \lim_{y\to\infty} \int_1^y \frac{dx}{x^2} = \lim_{y\to\infty} \left[-\frac{1}{x}\right]_1^y =$$

$$= \lim_{y\to\infty} -\frac{1}{y} + 1 = 1 - \lim_{y\to\infty} \frac{1}{y} = 1$$

Věta 8.22 (Integrální kriterium)

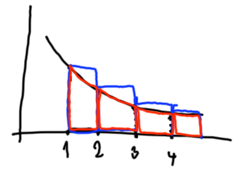
Nechť $f : [1, \infty) \longrightarrow \mathbb{R}$ je spojitá nezáporná klesající funkce



Potom řada $\sum f(n)$ konverguje, právě když existuje konečný integrál

$$\int_1^\infty f(x)\,dx.$$

Důkaz:



$$f(2)(2-1) \leq \int_1^2 f(x)\,dx \leq f(1)\cdot(2-1)$$

$$f(2) \leq \int_1^2 f(x)\,dx \leq f(1)$$

$$f(n) \leq \int_{n-1}^n f(x)\,dx \leq f(n-1)$$

$$\sum f(n) \leq \sum \int_{n-1}^n f(x)\,dx$$

$$S_n = \int_1^n f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \dots + \int_{n-1}^n$$

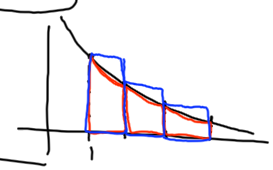
$$\downarrow$$

$$\int_1^\infty f(x)\,dx$$

$\int_1^\infty f(x)\,dx$ diverg.

$$\int_{n-1}^n f(x)\,dx \leq f(n-1)$$

$$\sum \int_{n-1}^n f(x)\,dx \leq \sum f(n-1)$$



Př. $\sum \frac{1}{n} = \infty$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{y\to\infty} \left[\ln x\right]_1^y = \lim_{y\to\infty} \ln y = \infty$$

$\sum \frac{1}{n^2}$ – konverguje

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{y\to\infty} \left[-\frac{1}{x}\right]_1^y = \lim -\frac{1}{y} + 1 = 1 < \infty$$