6.5 GONIOMETRICKÉ FUNKCE

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad \text{sinus}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{kosinus}$$

Věta 6.19

1. Funkce $\sin x, \cos x$ jsou spojité
2. funkce sin lichá, cos sudá.
3. $\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

$$1 = \cos^2(x) + \sin^2(x)$$

Důkaz: 1. plyne z Věty 6.13

2. $$\sin(-x) = \sum_{n=0}^{\infty} (-1)^n \frac{(-x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \underbrace{(-1)^{2n+1}}_{=-1} \frac{x^{2n+1}}{(2n+1)!}$$

$$= -\sin(x)$$

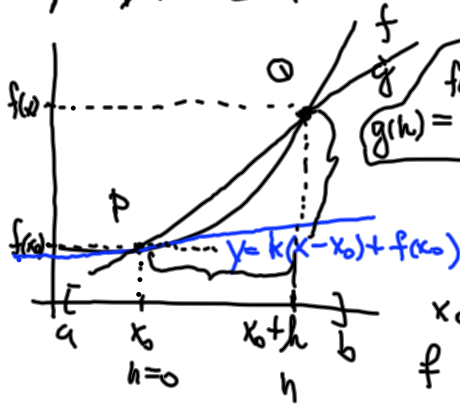
Tangens $\tan: \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi ; k \in \mathbb{Z}\} \to \mathbb{R}$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Kotangens $\cot: \mathbb{R} \setminus \{k\pi ; k \in \mathbb{Z}\} \to \mathbb{R}$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

7 DIFERENCIÁLNÍ POČET V $\mathbb{R}$



$$g(h) = \frac{f(x_0+h) - f(x_0)}{h}(x - x_0) + f(x_0)$$

$x_0 \in J$ derivace funkce f v bodě x_0, jako

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Příklad: $f(x) = c$.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

$f(x) = x$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{x_0 + h - x_0}{h} = 1$$

$f(x) = x^2$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0+h)^2 - x_0^2}{h} =$$

$$= \lim_{h \to 0} \frac{x_0^2 + 2x_0h + h^2 - x_0^2}{h} = \lim_{h \to 0} \frac{2x_0 + h}{1} = 2x_0$$

$f: J \to \mathbb{R}$ pokud $f'(x)$ $x \in X \subset J$ existuje a je konečná

$f': X \to \mathbb{R}$

$(c)' = 0$

$(x)' = 1$

$(x^2)' = 2x$

$(pow_n)' = n x^{n-1}$ $\quad (x^n)' = n \cdot x^{n-1} \quad \forall n \in \mathbb{N}$

$(1/x)' = -\frac{1}{x^2}$

$f(x) = \frac{1}{x}$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} =$$

$$= \lim_{h \to 0} \frac{\frac{x_0 - x_0 - h}{(x_0+h)x_0}}{h} = \lim_{h \to 0} \frac{\frac{-h}{(x_0+h)x_0}}{h} =$$

$$= \lim_{h \to 0} \frac{-1}{x_0^2 + x_0 h} = -\frac{1}{x_0^2}$$

Tečna ke grafu funkce $f: J \to \mathbb{R}$ $x_0 \in J$ $P = (x_0, f(x_0))$. Rovnice tečny ke grafu funkce f bodem P je přímka určená rovnicí

$$y = f'(x_0) \cdot (x - x_0) + f(x_0).$$

Title: 2. 27-3:35 odp. (3 of 5)

Derivace zleva, zprava

$$f'_-(x_0) = \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}$$

$$f'_+(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$$

Věta 7.1: Buď $f: J \longrightarrow \mathbb{R}$ a $x_0 \in J$, potom $f'(x_0)$ existuje právě, když existují $f'_-(x_0)$ a $f'_+(x_0)$ a rovnají se.

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$f(x) = |x|$

$$f'_-(0) = \lim_{h\to 0-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0-} \frac{|h|-|0|}{h} = \lim_{h\to 0-} \frac{|h|}{h} =$$

$$= \lim_{h\to 0-} \frac{-h}{h} = -1$$

$$f'_+(0) = \lim_{h\to 0+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0+} \frac{|h|-|0|}{h} = \lim_{h\to 0+} \frac{|h|}{h} =$$

$$= \lim_{h\to 0+} \frac{h}{h} = 1$$

Věta 7.2. Nechť $f: J \to \mathbb{R}$ $x_0 \in J$ a $f'(x_0)$ existuje a je konečná, potom je f v x_0 spojitá.

Důkaz: $\lim_{x\to x_0} f(x) = f(x_0) \Longleftrightarrow \lim_{x\to x_0} f(x) - f(x_0) = 0$

$$\lim_{x\to x_0} f(x) - f(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}(x-x_0) =$$

$$= \underbrace{\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}}_{f'(x_0)\in\mathbb{R}} \cdot \underbrace{\lim_{x\to x_0} x-x_0}_{0} = f'(x_0)\cdot 0 = 0$$

$$\operatorname{sgn}(x) = \begin{cases} 1 & x>0 \\ 0 & x=0 \\ -1 & x<0 \end{cases}$$

$f(x) = \operatorname{sgn}(x)$ $x_0 = 0$

$$f'_-(0) = \lim_{h\to 0-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0-} \frac{\operatorname{sgn}(h)}{h} =$$

$$= \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

$f'_+(0) = \ldots$ - - - - -

$$= \lim_{h\to 0^+} \frac{1}{h} = \infty$$

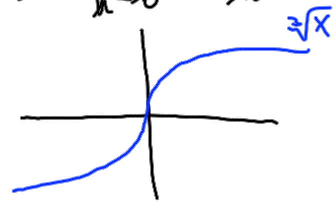
$\operatorname{sgn}'(0) = \infty$

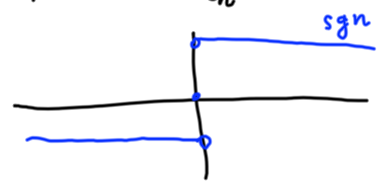
Title: 2. 27-3:50 odp. (4 of 5)

Příklad

$f(x) = \sqrt[3]{x}$ $\qquad$ $\underbrace{\sqrt[3]{x}}_{\text{spojitá funkce}}$ inverze k x^3

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{1}{h^{2/3}} = \infty$$





Věta 7.3. Nechť (f_n) je posloupnost funkcí $f_n : J \to \mathbb{R}$, f_n' existuje na J $\forall n \in \mathbb{N}$, $\sum f_n \to f$ a $\sum f_n' \rightrightarrows g$ na J. Potom platí existuje f' na J a

$$f' = g \quad \left(\text{neboli } f' = \sum f_n'\right).$$

Důsledek 7.4

1. $(e^x)' = e^x$;
2. $(\sin x)' = \cos x$;
3. $(\cos x)' = -\sin x$.

Důkaz 1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum f_n$

$$(e^x)' = e^x \qquad \begin{aligned} &0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \dots \\ &1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots = e^x \end{aligned}$$

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \dots$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \cos x$$

$\sin' x = \cos x$