Věta 7.5 Buďte $f, g: J \to \mathbb{R}$, $x_0 \in J$
existují $f'(x_0), g'(x_0)$ vlastní, $c \in \mathbb{R}$

1. $(f+g)'(x_0) = f'(x_0) + g'(x_0)$;
2. $(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + g'(x_0) f(x_0)$;
3. $(c \cdot f)'(x_0) = c \cdot f'(x_0)$.

Důkaz:

① $$\lim_{h \to 0} \frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x_0+h) + g(x_0+h) - f(x_0) - g(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} + \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h} =$$

$$= f'(x_0) + g'(x_0)$$

② $$\lim_{h \to 0} \frac{(f \cdot g)(x_0+h) - (f \cdot g)(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x_0+h)\, g(x_0+h) - f(x_0) \cdot g(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x_0+h) g(x_0+h) - f(x_0) g(x_0+h) + f(x_0) g(x_0+h) - f(x_0) g(x_0)}{h} =$$

$$= \lim_{h \to 0} \underbrace{\frac{f(x_0+h) - f(x_0)}{h}} g(x_0+h) + \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h} f(x_0) =$$

$$= \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \underbrace{\lim_{h \to 0} g(x_0+h)}_{g(x_0)} + g'(x_0) \cdot f(x_0) =$$

$$= f'(x_0) g(x_0) + g'(x_0) f(x_0)$$

③ $$\lim_{h \to 0} \frac{(c \cdot f)(x_0+h) - (c \cdot f)(x_0)}{h} = \lim_{h \to 0} \frac{c \cdot f(x_0+h) - c f(x_0)}{h} =$$

$$= c \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = c \cdot f'(x_0).$$

Př.: $f(x) = 3x^3 + 2x^2 + x$

$$f'(x) = 3 \cdot 3x^2 + 2 \cdot 2x + 1 = 9x^2 + 4x + 1$$

Věta 7.6 (Derivace složené funkce)
Nechť $f: J \to \mathbb{R}$, $g: I \to J$, $x_0 \in I$, existují vlastní derivace $f'(g(x_0))$ a $g'(x_0)$. Potom existuje $(f \circ g)'(x_0)$ a platí

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0).$$

Důkaz: $y_0 = g(x_0)$. Definuji $F: J \to \mathbb{R}$

$$F(y) = \begin{cases} \dfrac{f(y) - f(y_0)}{y - y_0} & y \neq y_0 \\ f'(y_0) & y = y_0 \end{cases}$$

spojitá v y_0

$$(y - y_0) F(y) = f(y) - f(y_0) \qquad \forall y \in J \qquad (*)$$

$$(g(x) - y_0) F(g(x)) = f(g(x)) - f(y_0)$$

$$\lim_{x \to x_0} \frac{(f \circ g)(x) - (f \circ g)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(g(x)) - f(\overbrace{g(x_0)}^{y_0})}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{(g(x) - g(x_0)) F(g(x))}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} \cdot \lim_{x \to x_0} F(g(x)) = g'(x_0) F(y_0) =$$

$$\lim_{x \to x_0} g(x) = g(x_0) = y_0$$

$$= g'(x_0) \cdot f'(g(x_0)).$$

Důsledek 7.7. Buďte $f, g : J \longrightarrow \mathbb{R}$ a $x_0 \in J$, $f'(x_0), g'(x_0)$ navíc $g'(x_0) \neq 0$, $g(J) \not\ni 0$. Potom

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\,g(x_0) - g'(x_0)\,f(x_0)}{g^2(x_0)} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

Důkaz:

$$\left(\frac{f}{g}\right)'(x_0) = \left(f \cdot \frac{1}{g}\right)'(x_0) = f'(x_0)\,\frac{1}{g(x_0)} + f(x_0)\left(\frac{1}{g}\right)'(x_0) =$$

$$= \frac{f'(x_0)}{g(x_0)} + f(x_0)\left(-\frac{1}{g^2(x_0)}\,g'(x_0)\right) = \frac{f'(x_0)\,g(x_0) - f(x_0)\,g'(x_0)}{g^2(x_0)}$$

Př.: $f(x) = \cos(2x) + \dfrac{3x^5}{e^{\sin x}}$

$$f'(x) = -\sin(2x)\cdot 2 + \frac{15x^4 e^{\sin x} - 3x^5 e^{\sin x}\cdot\cos x}{e^{2\sin x}}$$

$$a^x = e^{x \ln a}$$

$$f(x) = \boxed{x^{(e^x)}} = e^{e^x \cdot \ln x}$$

$$f'(x) = \underbrace{e^{e^x \cdot \ln x}}\cdot\left(e^x \ln x + e^x \cdot \ln' x\right) =$$

$$= x^{e^x}\cdot\left(e^x \ln x + e^x \cdot \frac{1}{x}\right)$$

$$e^x \cdot x^{e^x - 1}$$

$$x^{e^x}\cdot \ln(e^x)$$

Věta 7.8 (Derivace inverzní funkce)

Nechť $f: J \to I$ je spojitá rostoucí nebo klesající $x_0 \in J$, $f^{-1}: I \to J$ je její inverze. Pokud existuje vlastní $f'(x_0) \neq 0$, potom existuje

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)} \qquad \left((f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} \right)$$

Důkaz: $y_0 = f(x_0)$. f - spoj. rostoucí/klesající na intervalu homeomorfism I a J f^{-1} ex spojitá

$G: J \to \mathbb{R}$

$$G(x) = \begin{cases} \boxed{\dfrac{x - x_0}{f(x) - f(x_0)}} & x \neq x_0 \\ \dfrac{1}{f'(x_0)} & x = x_0 \end{cases}$$

$G(x_0)$ - spojitá

f - injektivní $x \neq x_0 \Leftrightarrow f(x) \neq f(x_0)$

$y \neq y_0$

$$\underline{\lim_{y \to y_0} G(f^{-1}(y))} = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{f(f^{-1}(y)) - f(f^{-1}(y_0))} =$$

$$= \underline{\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}} = \underline{(f^{-1})'(y_0)}$$

$$\underline{\lim_{y \to y_0} G(f^{-1}(y))} = G\left(\lim_{y \to y_0} f^{-1}(y)\right) = \underbrace{G(\underbrace{f^{-1}(y_0)}_{x_0}) = \frac{1}{f'(x_0)}}$$

Důsledek 7.9 Platí

1. $\ln'(x) = \frac{1}{x}$ $\quad x > 0$
2. $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$ $\quad x \in (-1, 1)$
3. $\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}$;
4. $\arctan'(x) = \frac{1}{x^2+1}$;
5. $\operatorname{arccotan}'(x) = -\frac{1}{x^2+1}$;
6. $\forall a > 0$ $(a^x)' = a^x \cdot \ln a$;
7. $\forall a \in \mathbb{R}$ $(x^a)' = a x^{a-1}$.

Důkaz: $f = \exp$ $\qquad (f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$

$f^{-1} = \ln$

$$\ln'(x) = \frac{1}{\exp'(\ln(x))} = \frac{1}{\exp(\ln(x))} = \frac{1}{x}$$

(4) $\quad f^{-1} = \arctan$, $f = \tan$

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)} \qquad \tan(x) = \frac{\sin x}{\cos x}$$

$$\tan'(x) = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\arctan'(x) = \frac{1}{\frac{1}{\cos^2(\operatorname{artg} x)}} = \frac{\sin^2(\arctan x)}{(\tan(\arctan(x)))^2} = \sin$$