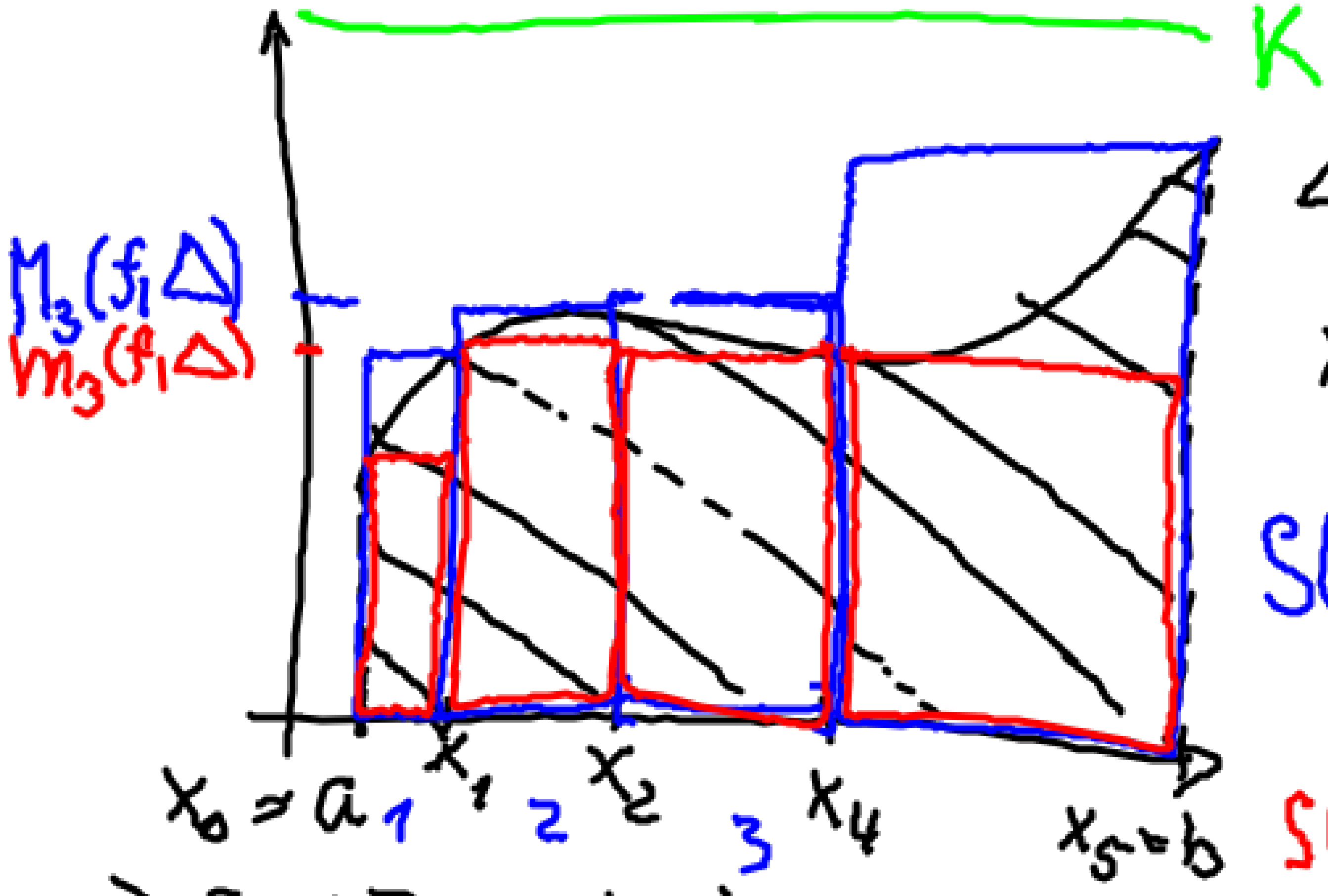


$$\Delta = (x_0 = a, x_1, x_2, \dots, x_n \overset{b}{=})$$

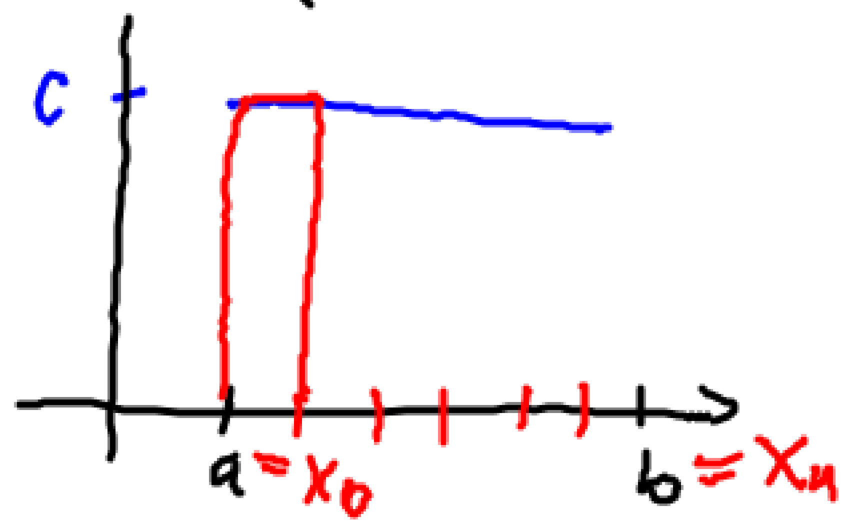
$$x_0 < x_1 < x_2 < \dots < x_n$$

$$S(f,\Delta) = \sum_{i=1}^{n} M_i(f,\Delta)(x_i - x_{i-1})$$

$$s(f,\Delta) = \sum_{i=1}^{n} m_i(f,\Delta)(x_i - x_{i-1})$$

$D[a,b]$ – množina všech dělení $[a,b]$

- $\overline{\int_a^b} f(x)\,dx = \inf\limits_{\Delta \in D[a,b]} S(f,\Delta) = \inf\{S(f,\Delta) \mid \Delta \in D[a,b]\}$
- $\underline{\int_a^b} f(x)\,dx = \sup\limits_{\Delta \in D[a,b]} s(f,\Delta)$
- $\int_a^b f(x)\,dx$



Příklad $\quad f(x) = c \qquad \int_a^b f(x)\,dx = c(b-a)$

$$\Delta = (x_0, x_1, \dots, x_n) \in D[a,b]$$

$$S(f,\Delta) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c\sum_{i=1}^{n} x_i - x_{i-1} =$$

$$= c(\cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \cancel{x_3} - \cancel{x_2} + \cancel{\dots} + \cancel{x_{n-1}} - \cancel{x_{n-2}} + x_n - \cancel{x_{n-1}})$$

$$= (b-a)c$$

$$s(f,\Delta) = c\sum_{i=1}^{n} x_i - x_{i-1} = c(b-a)$$

$$\underline{\int_a^b} f(x)\,dx = \sup_{\Delta \in D[a,b]} s(f,\Delta) = c(b-a)$$

$$\overline{\int_a^b} f(x)\,dx = \inf_{\Delta \in D[a,b]} S(f,\Delta) = c(b-a)$$

Lemma 8.1 Je-li Δ' zjemněním dělení Δ (Δ dělení $[a,b]$), potom

$$s(f,\Delta) \leq s(f,\Delta') \qquad S(f,\Delta) \geq S(f,\Delta')$$

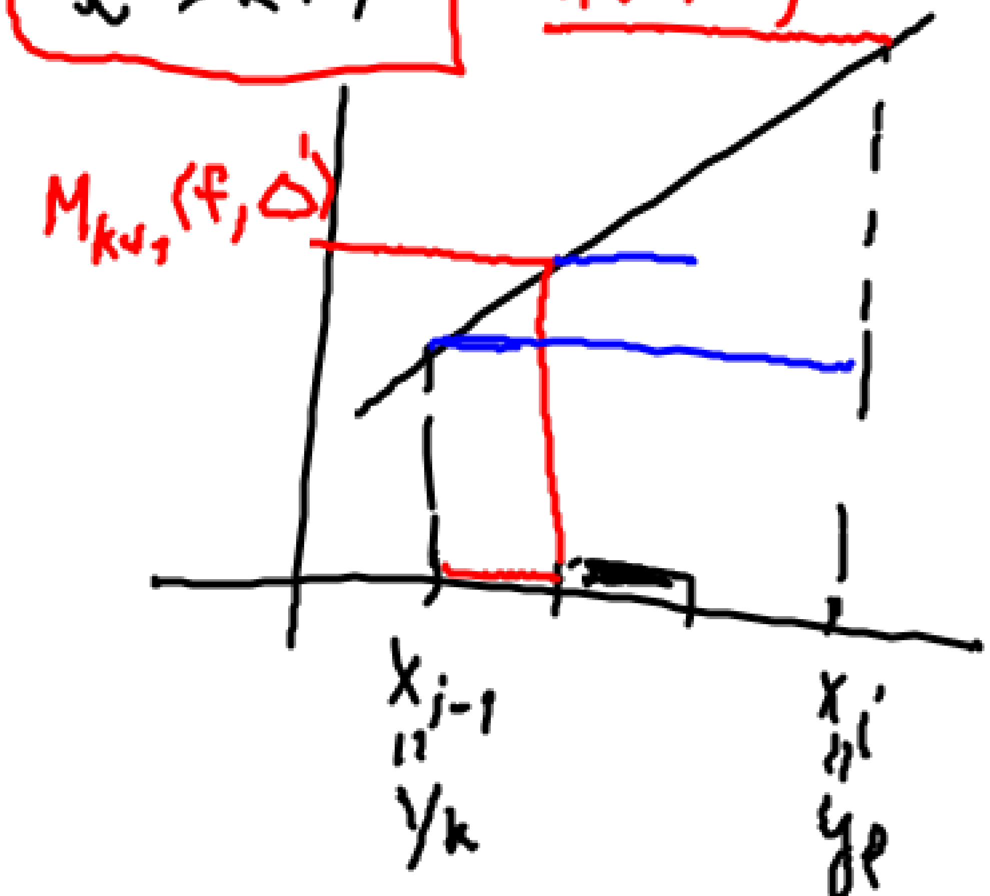
Důkaz: $\Delta' = (y_0, y_1, \dots, y_m)$ $\quad \Delta = (x_0, x_1, \dots, x_n)$

$$[x_{i-1}, x_i] = [y_k, y_l]$$

$l = k+1$

$$M_i(f,\Delta) = M_l(f,\Delta') \qquad m_i(f,\Delta) = m_l(f,\Delta')$$

$l > k+1$

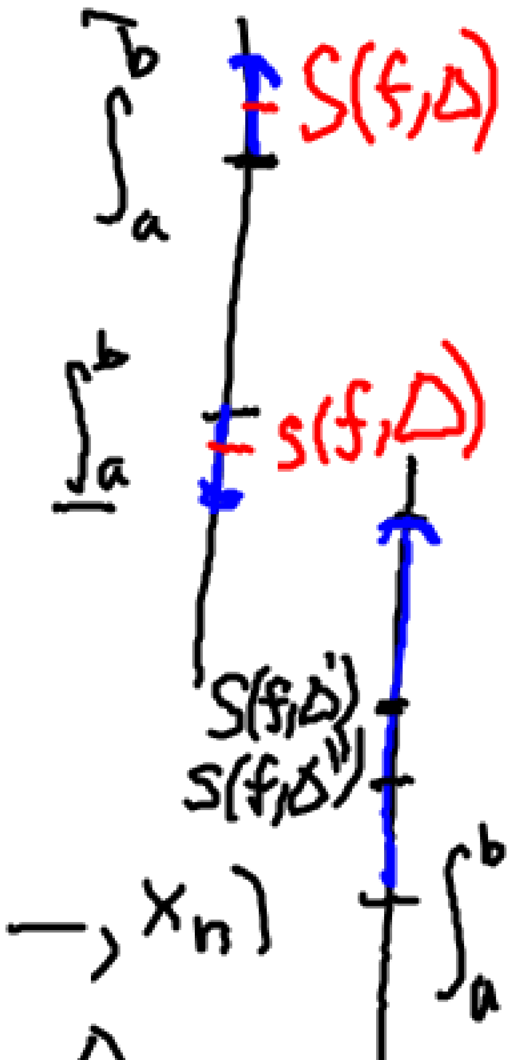


$$M_i(f,\Delta) \geq M_{k+1}(f,\Delta')$$
$$M_i(f,\Delta) \geq M_{k+2}(f,\Delta')$$
$$\vdots$$
$$M_i(f,\Delta) \geq M_l(f,\Delta')$$
$$m_i(f,\Delta) \leq M_{k+1}(f,\Delta')$$

$$M_i(f,\Delta)(x_i - x_{i-1}) \geq \sum_{j=k+1}^{l} M_j(f,\Delta')(y_j - y_{j-1})$$

$$S(f,\Delta) = \sum_{i=1}^{n} M_i(f,\Delta)(x_i - x_{i-1}) \geq \sum_{i=1}^{m} M_i(f,\Delta')(y_i - y_{i-1}) = S(f,\Delta')$$

Lemma 8.2 $f:[a,b]\to\mathbb{R}$ omezená $|f|<K$
ke každému $\varepsilon>0$ existuje $\delta>0$ takové, že je-li
$\Delta\in D[a,b]$ s $|\Delta|<\delta$ potom



$$\left[S(f,\Delta)-\int_a^b f(x)\,dx<\varepsilon,\right]$$

$$\int_a^b f(x)\,dx - s(f,\Delta)<\varepsilon.$$

Důkaz: $\varepsilon>0$ $\Delta'=(y_0,y_1,\dots,y_p)$

$$S(f,\Delta')-\int_a^b f(x)\,dx<\frac{\varepsilon}{2}$$

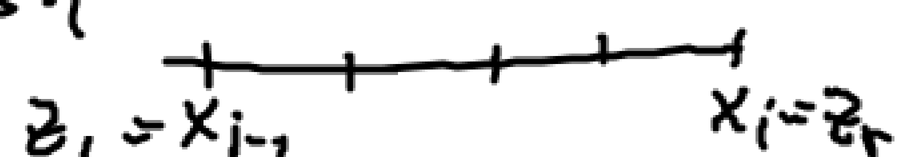
položíme $\delta=\frac{\varepsilon}{4Kp}$ $\Delta=(x_0,x_1,\dots,x_n)$

Vezmeme Δ'' společné zjemnění Δ',Δ

$$S(f,\Delta'')\le S(f,\Delta') \qquad S(f,\Delta'')\le S(f,\Delta)$$

$(x_{i-1},x_i)=(z_s,z_r)$ $\Delta''=(z_0,z_1,\dots,z_m)$

$(S(f,\Delta)-S(f,\Delta''))$ $s-r>1$



$$\left|\sum_{k=r+1}^{s} M_k(f,\Delta'')(z_{k-1}-z_k)\right|\le K\sum_{k=r+1}^{s} z_{k-1}-z_k=\underbrace{K(x_{i-1}-x_i)}_{<K\delta}$$

$$s(f,\Delta)-S(f,\Delta'')<pK\delta=\frac{pK\varepsilon}{4Kp}=\frac{\varepsilon}{4}$$

Věta 8.3 1. Integrál $\int_a^b f(x)\,dx$ existuje potom pro každou posloupnost dělení (Δ_n) $\lim\limits_{n\to\infty} |\Delta| = 0$ platí

$$\lim_{n\to\infty} s(f,\Delta_n) = \lim_{n\to\infty} S(f,\Delta_n) = \int_a^b f(x)\,dx. \quad (*)$$

2. Jestliže (Δ_n) posl. dělení $\lim |\Delta_n| = 0$ a platí $(*)$ potom f je integrovatelná a $\lim S(f,\Delta_n) = \int_a^b f(x)\,dx$.

Důkaz 1. $\varepsilon > 0$ podle Lemmatu 8.2 $\exists \delta > 0$ pro Δ $|\Delta| < \delta$

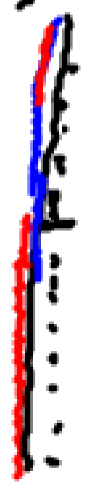
$$\left|S(f,\Delta) - \int_a^b f(x)\,dx\right| < \varepsilon$$

existuje číslo n_0 : $\forall n > n_0$ $|\Delta_n| < \delta$

2.

$$\underbrace{\inf\{S(f,\Delta) \mid \Delta \in D[a,b]\}}_{S_n(f,\Delta_n)} \geq \underbrace{\sup\{s(f,D) \mid \Delta \in D[a,b]\}}_{s(f,\Delta_n)}$$

$$S_n(f,\Delta_n) \longrightarrow \bullet \longleftarrow s(f,\Delta_n)$$

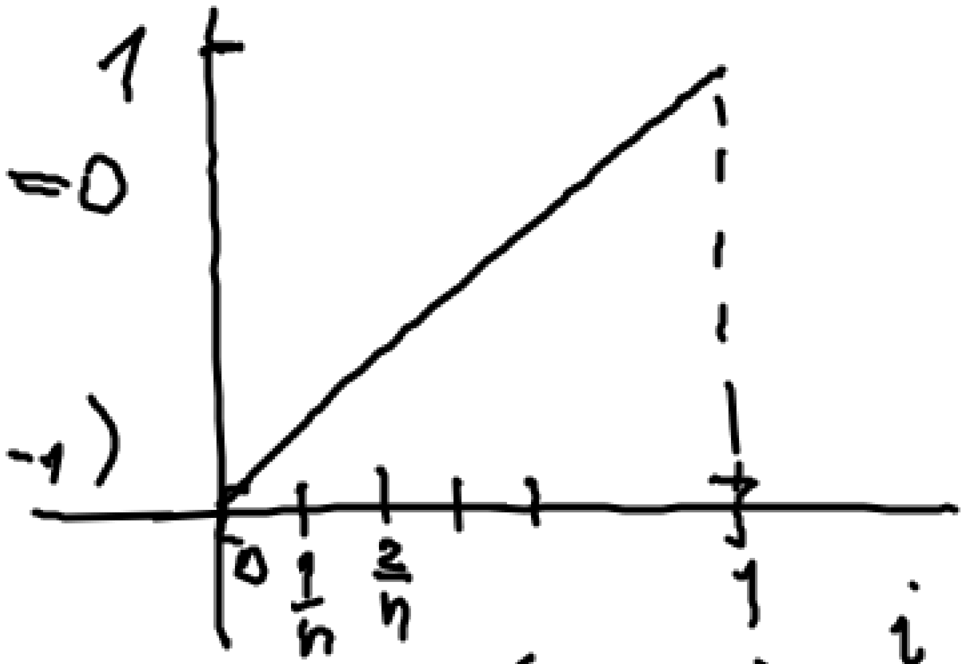


$f(x) = x$

$\Delta_1 = (0, 1)$ $\quad \lim |\Delta n| = 0$

$\Delta_2 = (0, \frac{1}{2}, 1)$

$\Delta_3 = (0, \frac{1}{3}, \frac{2}{3}, 1)$ $\quad (x_i - x_{i-1})$



$M_i(f, \Delta n) = \frac{i}{n}$

$m_i(f, \Delta n) = \frac{i-1}{n}$

$$S(f, \Delta n) = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \cdot \frac{n(1+n)}{2} = $$

$$= \frac{1+n}{2n} = \frac{1}{2} + \frac{1}{2n}$$

$$s(f, \Delta n) = \sum_{i=1}^{n} \frac{i-1}{n} \frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i-1 = \frac{1}{n^2} \cdot \frac{n(0+n-1)}{2} = $$

$$= \frac{n-1}{2n} = \frac{1}{2} - \frac{1}{2n}$$

$$\lim S(f, \Delta n) = \lim \frac{1}{2} + \frac{1}{2n} = \frac{1}{2}$$

$$\lim s(f, \Delta n) = \lim \frac{1}{2} - \frac{1}{2n} = \frac{1}{2},$$

$f : [a, b] \to \mathbb{R} \quad : \quad f(x) = 0 \quad x \neq \{c_1, c_2, \dots, c_n\}$

$\int_a^b f(x)\,dx = 0$