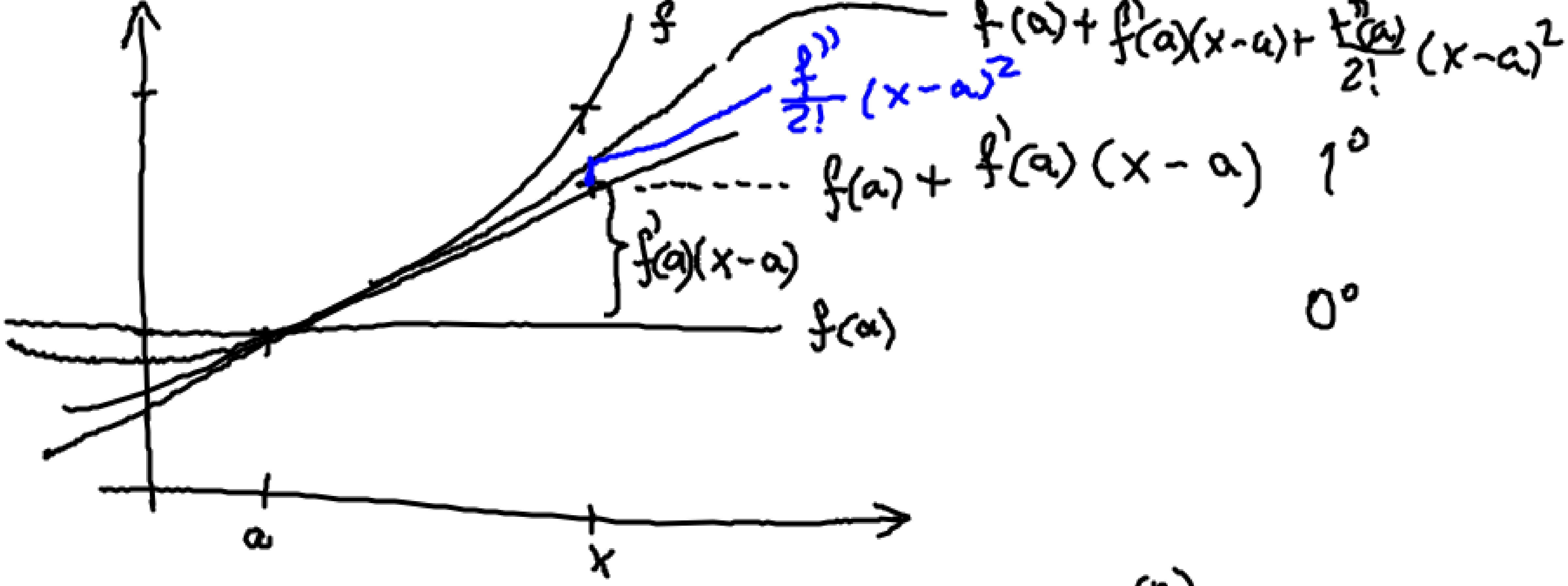


$f: X \to \mathbb{R}$ $\quad X \supset [a,b]$ má $f^{(n)}(a)$

Taylorův polynom $P: \mathbb{R} \to \mathbb{R}$ stupně $n \in \mathbb{N}$ se středem v bodě a.

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$
$$\dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Věta 7.23 (Taylorova)

Buďte $a, x \in \mathbb{R}$ $I = [a, x]$, $f : I \to \mathbb{R}$ která má v I derivace až do řádu $n+1$ a nechť $\varphi : I \to \mathbb{R}$ má uvnitř I nenulovou derivaci

Potom existuje $\xi \in (a, x)$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x) \qquad a < \xi < x$$

kde

$$R_{n+1}(x) = \frac{(x-\xi)^n}{n!} \frac{\varphi(x) - \varphi(a)}{\varphi'(\xi)} f^{(n+1)}(\xi).$$

Důkaz:

$$F(t) = f(x) - f(t) - f'(t)\frac{x-t}{1!} - f''(t)\frac{(x-t)^2}{2!} - \dots - f^{(n)}(t)\frac{(x-t)^n}{n!}$$

$$F'(t) = -\cancel{f'(t)} - \left(\cancel{f''(t)\frac{x-t}{1!}} - \cancel{f'(t)}\right) - \left(f'''(t)\frac{(x-t)^2}{2!} - \cancel{f''(t)\frac{2(x-t)}{2!}}\right)$$

$$- \cancel{\dots} - \left(f^{(n+1)}(t)\frac{(x-t)^n}{n!} - \cancel{f^{n}(t)\frac{n(x-t)^{n-1}}{n!}}\ (n-1)!\right)$$

$$= -f^{(n+1)}(t)\frac{(x-t)^n}{n!}$$

F, φ

existuje $\boxed{\xi \in (a, x)}$

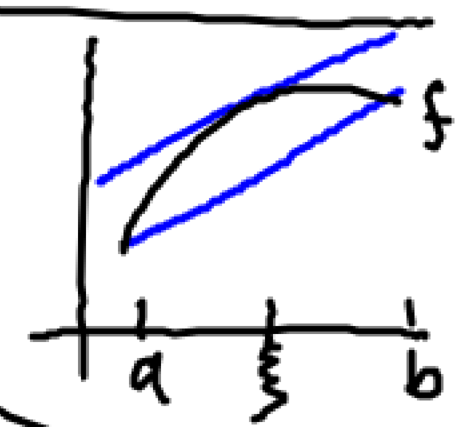
$$\frac{F(x)-F(a)}{\varphi(x)-\varphi(a)} = \frac{F'(\xi)}{\varphi'(\xi)} =$$

$$= -\frac{f^{(n+1)}(\xi)}{n!}\,\frac{(x-\xi)^n}{\varphi'(\xi)}$$

$f, g \quad [a,b] \quad \xi \in (a,b)$

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

V 7.15



$0-(x-a)^{n+1}$

$\varphi'(t) = -(n+1)(x-t)^n$

$\varphi'(\xi) = -(n+1)(x-\xi)^n$

$\varphi'(\xi) = 1$

$$F(x)-F(a) = 0 - R_{n+1}(x)$$

$$+\frac{R_{n+1}(x)}{\varphi(x)-\varphi(a)} = +\frac{f^{(n+1)}(\xi)}{n!}\,\frac{(x-\xi)^n}{\varphi'(\xi)}$$

$$R_{n+1}(x) = f^{(n+1)}(\xi)\,\frac{\varphi(x)-\varphi(a)}{n!}\,\frac{(x-\xi)^n}{\varphi'(\xi)}$$

$\varphi(t) = (x-t)^{n+1}$

$$R_{n+1}(x) = \frac{(x-a)^{n+1}}{(n+1)!}\,f^{(n+1)}(\xi) \quad \boxed{\text{Lagrange}}$$

$\varphi(t) = t$

$$R_{n+1}(x) = \frac{(x-\xi)^n(x-a)}{n!}\,f^{(n+1)}(\xi) \quad \boxed{\text{Cauchy}}$$

e ..... exp(1)   0,01

$$\exp^{(n)}(x) = \exp(x)$$

$$\exp^{(n)}(0) = 1$$

$$[a, x] = [0, 1]$$

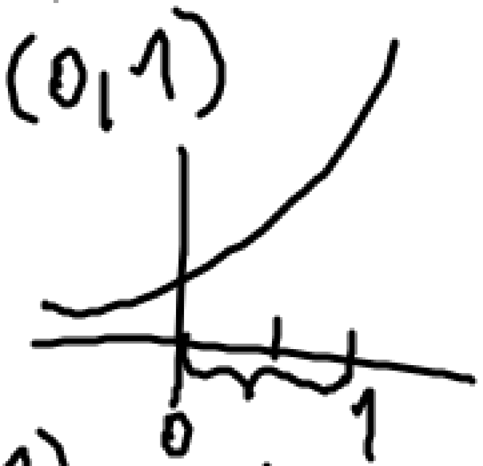
$$P(x) = \exp(0) + \frac{\exp'(0)}{1!}(x-0) + \frac{\exp''(0)}{2!}(x-0)^2 + \ldots$$

$$\ldots + \frac{\exp^{(n)}(0)}{n!}(x-0)^n =$$

$$= \boxed{1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}}$$

$\exp(1) < 4$

$\xi \in (0, 1)$



$$R_{n+1}(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

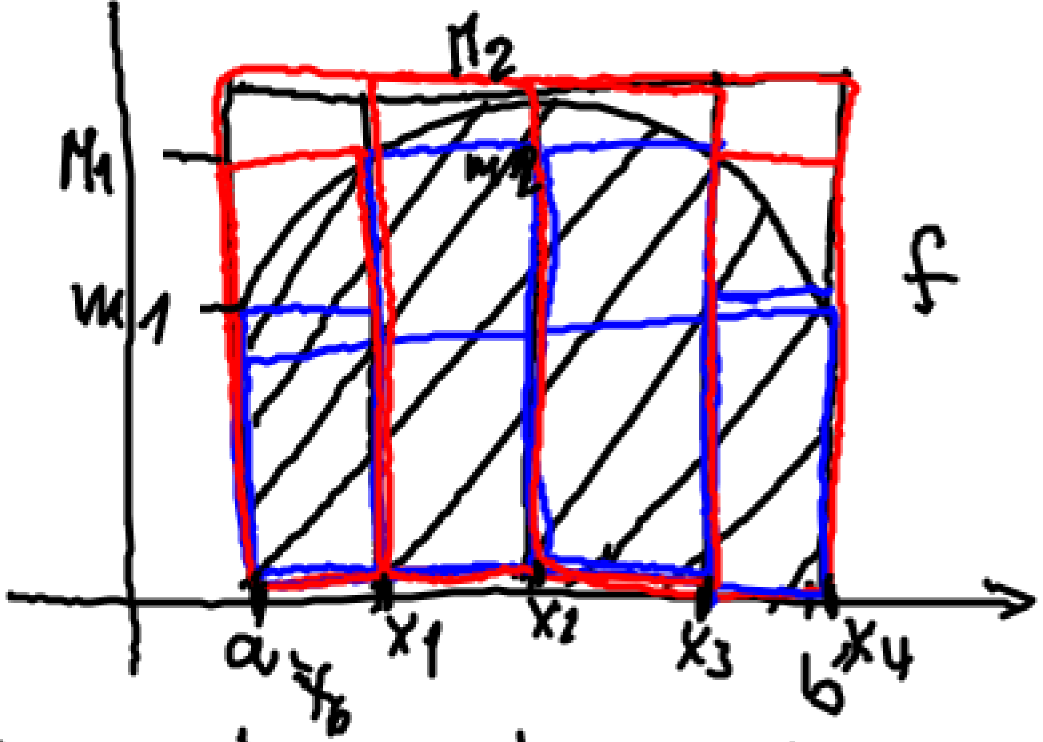
$$R_{n+1}(1) = \frac{1}{(n+1)!} \exp(\xi) < \frac{\exp(1)}{(n+1)!} < \frac{4}{(n+1)!} < 0{,}01$$

$n = 4$ $\frac{4}{5!} = \frac{4}{120} = \frac{1}{30}$

$n = 5$ $\frac{4}{6!} = \frac{4}{720} = \frac{1}{180}$

$$P(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$$

$$= \frac{120 + 120 + 60 + 20 + 5 + 1}{120} = \frac{326}{120} = 2{,}716$$



Dělení intervalu $[a,b]$

$\Delta = (x_0, x_1, \ldots, x_n)$

$x_0 = a,\ x_n = b$

$x_0 < x_1 < x_2 < \ldots < x_n$

Δ, Δ' $\quad \Delta'$ je zjemnění dělení Δ

Δ', Δ'' společné zjemnění Δ'''

$\Delta = (x_0, x_1, \ldots, x_n)$ $\qquad D[a,b]$

$$M_i(f,\Delta) = \sup_{x \in (x_{i-1}, x_i)} f(x)$$

$$m_i(f,\Delta) = \inf_{x \in (x_{i-1}, x_i)} f(x)$$

$$S(f,\Delta) = \sum_{i=1}^{n} M_i(f,\Delta) \cdot (x_i - x_{i-1})$$ – Horní součet

$$s(f,\Delta) = \sum_{i=1}^{n} m_i(f,\Delta)(x_i - x_{i-1})$$ – Dolní součet

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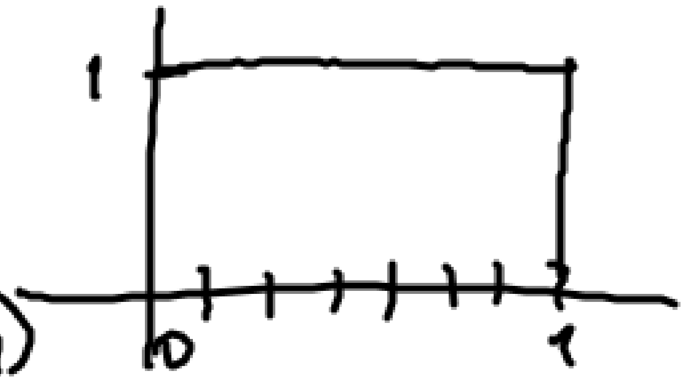
$$\overline{\int_a^b} f(x)\,dx = \inf_{\Delta \in D[a,b]} S(f,\Delta)$$ Horní integrál

$$\underline{\int_a^b} f(x)\,dx = \sup_{\Delta \in D[a,b]} s(f,\Delta)$$ Dolní integrál

$f:[0,1]$ $\quad f(x)=1$



$\Delta=(x_0,\dots,x_n)$

$S(f,\Delta)=\sum_{i=1}^{n} M_i(x_i-x_{i-1})$

$=\sum_{i=1}^{n}(x_i-x_{i-1}) = x_n-\cancel{x_{n-1}}+\cancel{x_{n-1}}-\cancel{x_{n-2}}+\dots+\cancel{x_2}-\cancel{x_1}+\cancel{x_1}-x_0 = x_n-x_0$

$=1-0$

$=1$

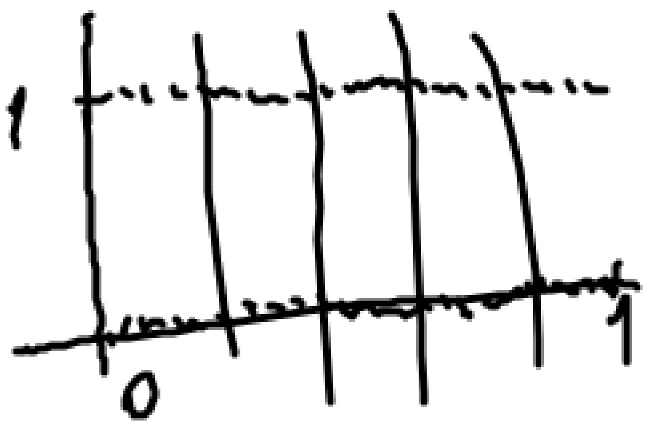
$s(f,\Delta)=\sum_{i=1}^{n} m_i(x_i-x_{i-1}) = \dots = 1$

$\overline{\int_a^1} 1\,dx = 1$

$\underline{\int_a^b} 1\,dx = 1$

$$\int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$$

$\Delta=(x_0,x_1,\dots,x_n)$ $\quad [0,1]$



$M_i(\chi,\Delta)=1$

$m_i(\chi,\Delta)=0$

$S(f,\Delta)=1$, $s(f,\Delta)=0$

$\overline{\int_a^b}\chi(x)\,dx = 1$, $\quad \underline{\int_a^b}\chi(x)\,dx=0.$