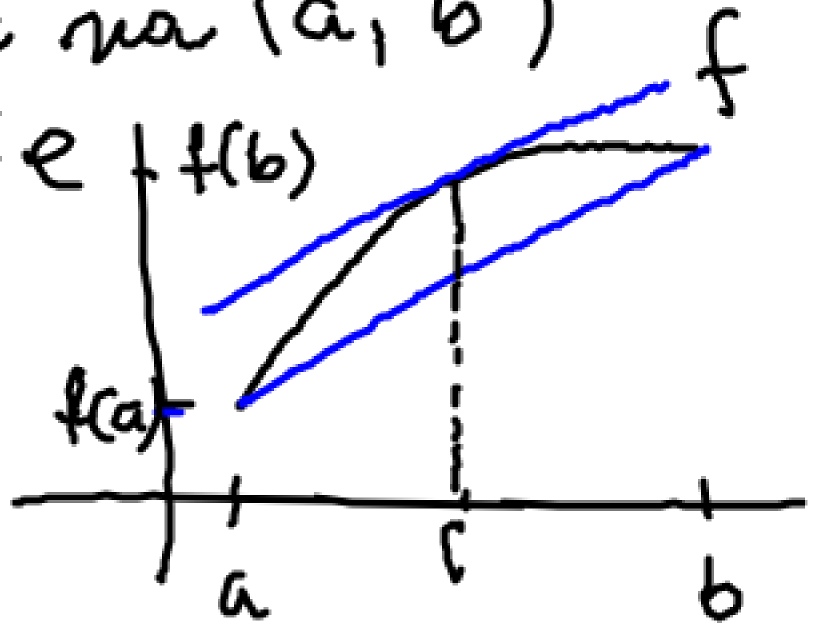
Věta (o střední hodnotě)

$f:[a,b]\longrightarrow\mathbb{R}$ f' existuje na (a,b)

existuje $c\in(a,b)$ takový, že

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



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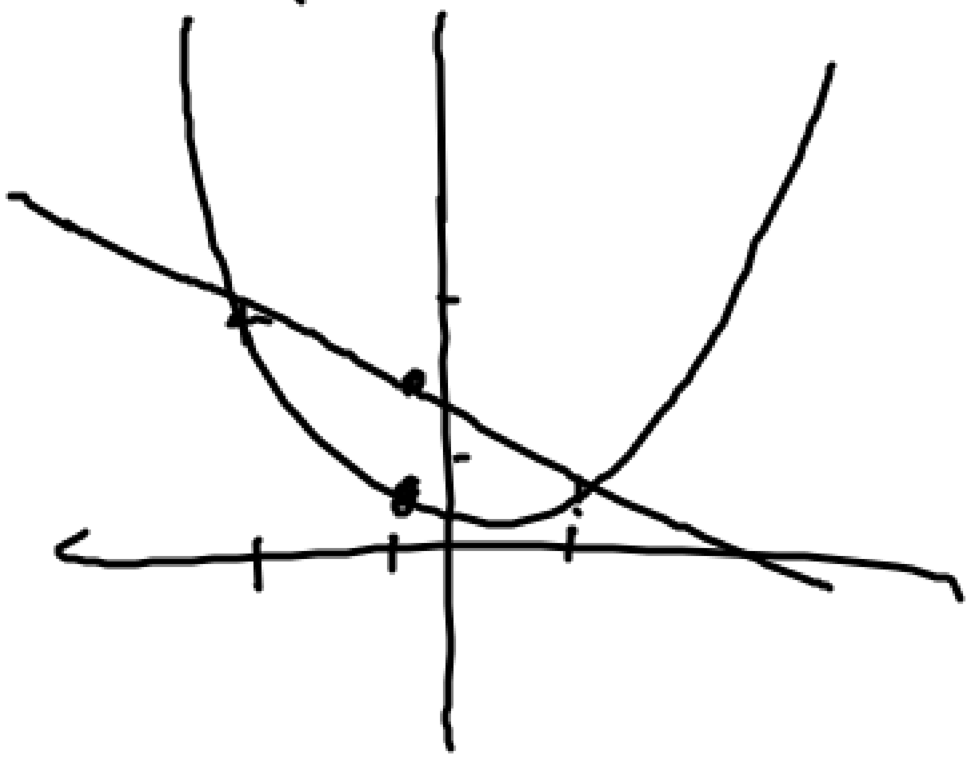
f je na intervalu $I\subset\mathbb{R}$ konvexní

jestliže $x<y<z$

$$f(x)(z-y)+f(y)(x-z)+f(z)(y-x)>0$$



Věta 7.18 $f: J \longrightarrow \mathbb{R}$

1. Je-li $f''(x) > 0$ pro $x \in (a,b)$, potom f na J konvex
2. Je-li $f''(x) < 0$ pro $x \in (a,b)$, potom f na J konkávní.

Důkaz: $x, y, z \in (a,b)$ $\quad \underline{x < y < z}$

$c \in (x,y) \quad f'(c) = \dfrac{f(y) - f(x)}{y - x} \quad f'(c)(y-x) = f(y) - f(x)$

$d \in (y,z) \quad f'(d) = \dfrac{f(z) - f(y)}{z - y} \quad f'(d)(z-y) = f(z) - f(y)$

$$\big(f'(d) - f'(c)\big)\underbrace{(y-x)}_{>0}\underbrace{(z-y)}_{>0} = \big(f(z) - f(y)\big)(y-x)$$

$$- \big(f(y) - f(x)\big)(z-y) =$$

$$= f(z)y - f(z)x - \cancel{f(y)y} + f(y)x - f(y)z + \cancel{f(y)y} + f(x)z - f(x)y =$$

$$= f(x)(z-y) + f(y)(x-z) + f(z)(y-x)$$

1. $f'' > 0 \quad f'$ rostoucí $\quad f'(d) > f'(c) \quad \to > 0$
2. $f'' < 0 \quad f'(d) < f'(c) \quad \to < 0$

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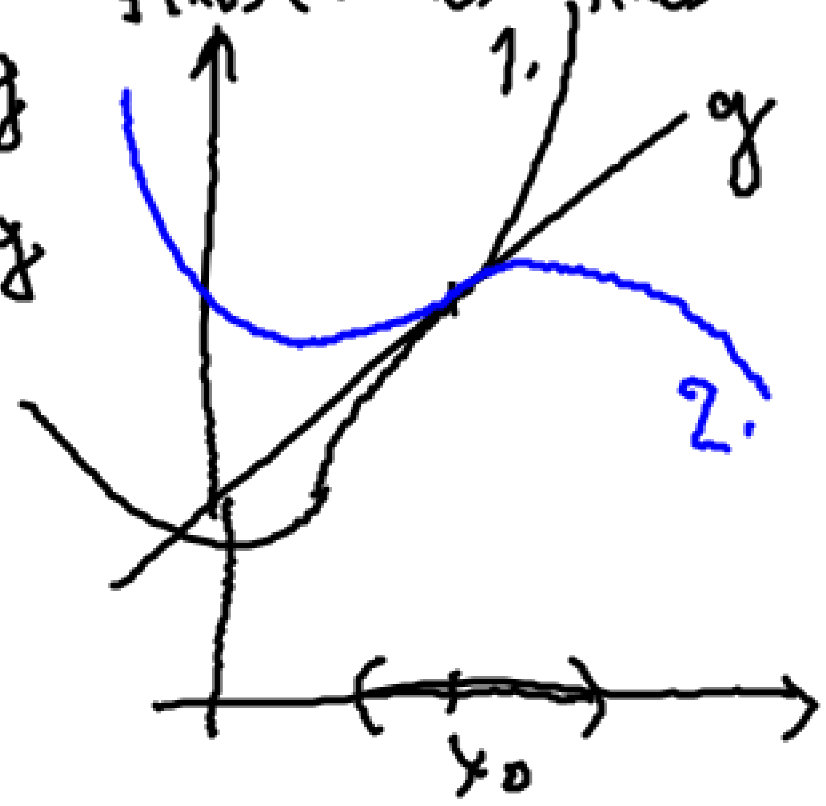
f má v x_0 inflexi $\exists (x_0 - \delta, x_0 + \delta)$

$f'(x_0)$ existuje vlastní $\quad g(x) = f'(x_0)(x - x_0) + f(x_0)$

1. $f|_{(x_0-\delta, x_0)} < g$ a $f|_{(x_0, x_0+\delta)} > g$
2. $f|_{(x_0-\delta, x_0)} > g$ a $f|_{(x_0, x_0+\delta)} < g$



Věta 7.19 $f: J \to \mathbb{R}$ a $\exists n \in \mathbb{N}$ takové, že $f'(x_0) = 0, f''(x_0) = 0, \dots, f^{(n-1)}(x_0) = 0, f^{(n)}(x_0) \neq 0$

1. Je-li n liché a $f^{(n)}(x_0) > 0$, pak f je v x_0 rostoucí
2. Je-li n liché a $f^{(n)}(x_0) < 0$, pak f je v x_0 klesající
3. sudé a $f^{(n)}(x_0) > 0$, pak f má v x_0 lok. minimum
4. sudé a $f^{(n)}(x_0) < 0$, pak f lok. maximum

Důkaz: Je-li $n = 1$ Věta 7.11 f v x_0 rostoucí

$n \in \mathbb{N}$

(i) $n+1$ je sudé to znamená n je liché

$\underbrace{g = f'}_{=0}$, $\underbrace{g' = f''}_{=0}$, $\dots$, $\underbrace{g^{(n-2)} = f^{(n-1)}}_{=0}$, $\underbrace{g^{(n-1)} = f^{(n)}}_{=0}$, $\underbrace{g^{(n)} = f^{(n+1)}}_{\neq 0}$

podle 7.19 g je rostoucí f' rostoucí

$f'(x_0) = 0$

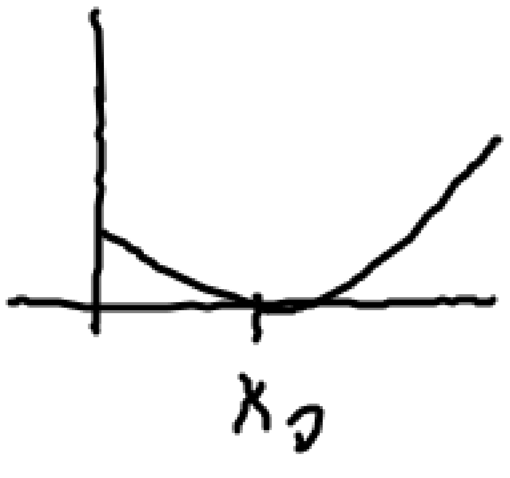
(ii) $n+1$ liché n je sudé $g = f'$

podle 7.19 g má v x_0 lokální minimum

f' má v x_0 lok. minimum $f' > 0$

$f'(x_0) = 0$

f je v x_0 rostoucí

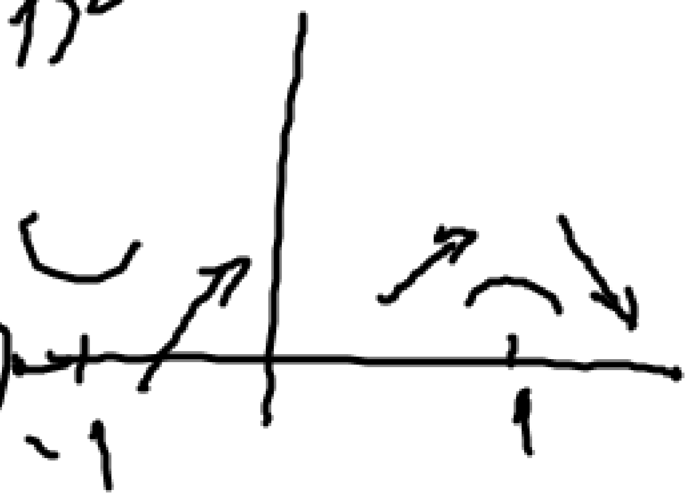


Důsledek 7.20 $f: J \to \mathbb{R}$ $\exists n \in \mathbb{N}$:
$f'(x_0)=0, \dots, f^{(n-1)}(x_0)=0$, $f^{(n)}(x_0)\neq 0$

1. Je-li n sudé $f^{(n)}(x_0)>0$ f je na okolí x_0 konvexní
2. Je-li n sudé $f^{(n)}(x_0)<0$ konkávní
3. Je-li n liché f má v x_0 inflexi.

Příklad $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \frac{x}{x^2+1}$

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} = 0 \qquad x = \pm 1$$



$$f''(x) = \frac{-2x(x^2+1)^2 - 2(x^2+1)2x(1-x^2)}{(x^2+1)^4}$$

$$= \frac{-2x(x^2+1) - 4x(1-x^2)}{(x^2+1)^3} = -2x\,\frac{x^2+1+2-2x^2}{(x^2+1)^3} =$$

$$= -2x\,\frac{3-x^2}{(x^2+1)^3} = 0 \qquad x=0,\quad x=\pm\sqrt{3}$$

Věta 7.21 (L'Hospitalovo pravidlo)

Buďte $f, g : J \to \mathbb{R}$ $x_0 \in \overline{\mathbb{R}}$ $\lim_{x\to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$

Pokud existuje limita $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ pak existuje $\lim_{x\to x_0} \frac{f(x)}{g(x)}$

a platí

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

$$\bar{f}(x) = \begin{cases} f(x) & x \neq x_0 \\ 0 & x = x_0 \end{cases}$$

$$\bar{g}(x) = \begin{cases} g(x) & x \neq x_0 \\ 0 & x \neq x_0 \end{cases}$$

Důkaz: $\lim_{x\to x_0+}$ $x > x_0$

$x_0 < c < x$ $[x_0, x]$

$$\frac{\bar{f}'(c)}{\bar{g}'(c)} = \frac{\bar{f}(x) - \bar{f}(x_0)}{\bar{g}(x) - \bar{g}(x_0)} = \frac{\bar{f}(x)}{\bar{g}(x)} = \frac{f(x)}{g(x)}$$

$$\lim_{x\to x_0+} \frac{f'(x)}{g'(x)} = \lim_{x\to x_0+} \frac{f(x)}{g(x)}$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} \underset{L'H}{=} \lim_{x\to 0} \frac{\cos(x)}{1} = 1$$

$$\lim_{x\to\infty} \frac{e^x}{x^n} \underset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{n x^{n-1}} \underset{L'H}{=} \text{—} = \lim_{x\to\infty} \frac{e^x}{n!} = \infty$$

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$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{1}{\ln(x)}\right)=\lim_{x\to 1}\frac{\ln(x)-x+1}{(x-1)\ln(x)}=$$

$$\overset{L'H}{=}\lim_{x\to 1}\frac{\frac{1}{x}-1}{\ln(x)+\frac{x-1}{x}}=\lim_{x\to 1}\frac{\frac{1-x}{x}}{\frac{x\ln(x)+x-1}{x}}=\lim_{x\to 1}\frac{1-x}{x\ln x+x-1}$$

$$\overset{L'H}{=}\lim_{x\to 1}\frac{-1}{\ln(x)+\frac{x}{x}+1}=\lim_{x\to 1}\frac{-1}{\ln(x)+2}=-\frac{1}{2}$$

$$\lim_{x\to 0}x^{1/x}=\lim_{x\to 0}e^{\frac{1}{x}\ln(x)}=\exp\left(\lim_{x\to 0}\frac{\ln(x)}{x}\right)$$