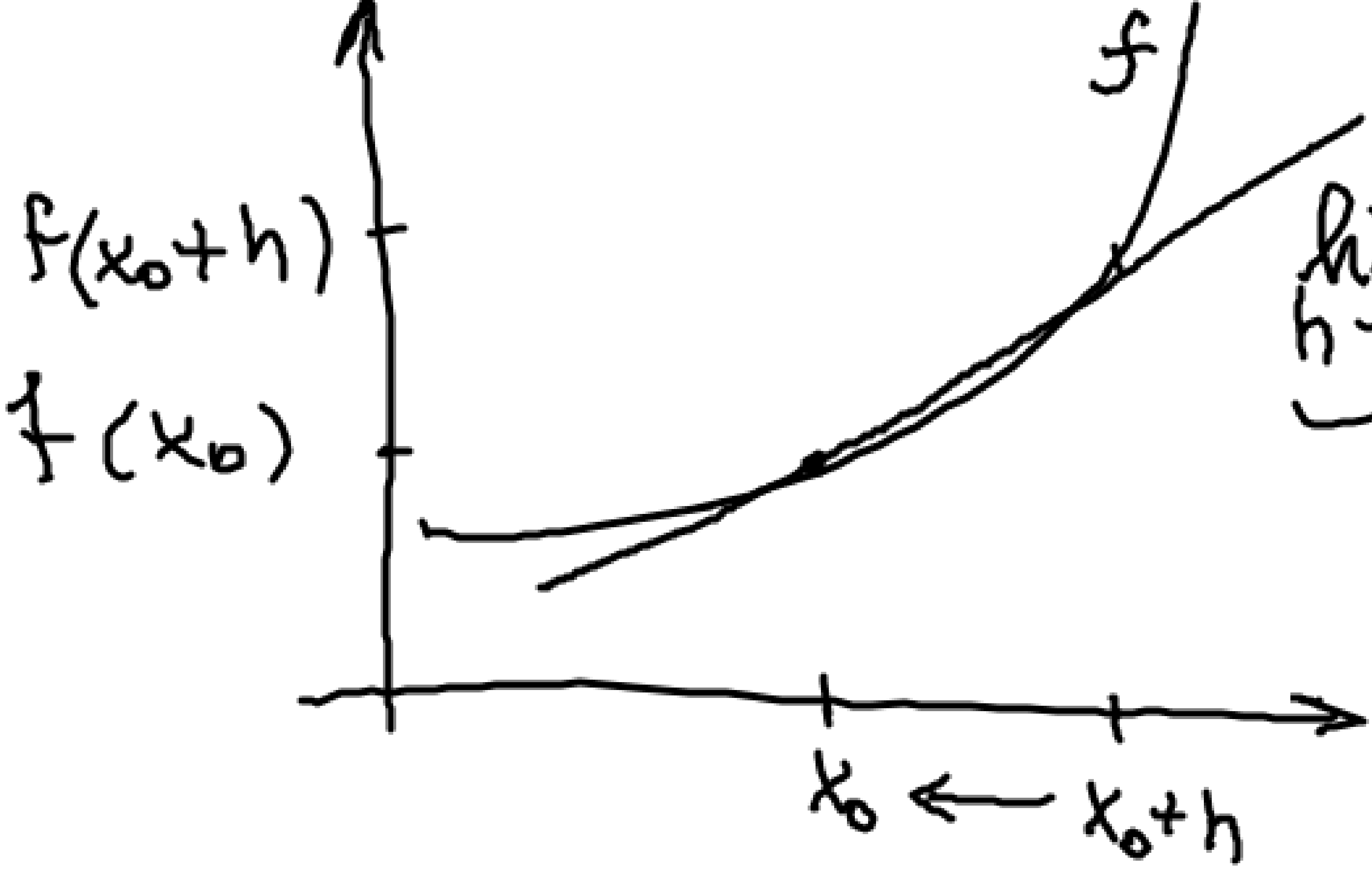

$$\underbrace{\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}}_{=\,f'(x_0)}$$

Věta 7.5 Buďte $f, g : J \longrightarrow \mathbb{R}$, $x_0 \in J$ existují vlastní $f'(x_0), g'(x_0)$ potom

1. $(f+g)'(x_0) = f'(x_0) + g'(x_0)$;
2. $(f \cdot g)'(x_0) = f'(x_0)\,g(x_0) + g'(x_0)\,f(x_0)$

Důkaz

1. $$(f+g)'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)+g(x_0+h)-f(x_0)+g(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h} + \lim_{h \to 0} \frac{g(x_0+h)-g(x_0)}{h} = f'(x_0) + g'(x_0)$$

2. $$(f \cdot g)'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)\cdot g(x_0+h) - f(x_0)\,g(x_0) + g(x_0+h)f(x_0) - g(x_0+h)\,f(x_0)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x_0+h)\,g(x_0+h) - g(x_0+h)\,f(x_0)}{h} + \lim_{h \to 0} \frac{g(x_0+h)f(x_0) - f(x_0)\,g(x_0)}{h} =$$

$$\underbrace{\lim_{h \to 0} g(x_0+h)}_{g(x_0)} \cdot \underbrace{\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}}_{f'(x_0)} + f(x_0) \cdot \underbrace{\lim_{h \to 0} \frac{g(x_0+h)-g(x_0)}{h}}_{g'(x_0)}$$

$$g(x_0) \cdot f'(x_0) + f(x_0) \cdot g'(x_0)$$

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3. $(c f')(x_0) = c f'(x_0) \quad c \in \mathbb{R}$

Důk: $(c f)'(x_0) = \underbrace{c'}_{0} \cdot f(x_0) + c \cdot f'(x_0) = c f'(x_0).$

Věta 7.6 (Derivace složené funkce)

$f : J \to \mathbb{R}$, $g : I \to J$, $x_0 \in I$ existují $f'(g(x_0))$, $g'(x_0)$.

potom existuje vlastní $(f\circ g)'(x_0)$

$$(f\circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0).$$

Důkaz $\quad F : J \to \mathbb{R}$

$y_0 = g(x_0)$

$$F(y) = \begin{cases} \dfrac{f(y)-f(y_0)}{y-y_0} & y \neq y_0 \\ f'(y_0) & y = y_0 \end{cases}$$

$$F(y)\cdot(y-y_0) = f(y) - f(y_0)$$

$$(f\circ g)'(x_0) = \lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{x-x_0} \overset{(*)}{=} \lim_{x\to x_0} \frac{F(g(x))\cdot(g(x)-g(x_0))}{x-x_0} =$$

$$= \lim_{x\to x_0} F(g(x)) \cdot \underbrace{\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}}_{g'(x_0)} = F\underbrace{\left(\lim_{x\to x_0} g(x)\right)}\cdot g'(x_0)$$

$$= F(\overbrace{g(x_0)}^{y_0}) \cdot g'(x_0) = f'(g(x_0)) \cdot g'(x_0).$$

Důsledek $f, g : J \to \mathbb{R}$ $f'(x_0), g'(x_0)$, $g(x) \neq 0$ $\forall x \in J$

$\left(\frac{f}{g}\right)'(x_0)$ existuje a platí

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\cdot g(x_0) - g'(x_0) f(x_0)}{g^2(x_0)}$$

$$\left(\frac{f}{g}\right)(x) = f(x)\cdot\frac{1}{g(x)} = f(x) h(g(x))$$

$$h(x) = \frac{1}{x} \qquad \left(\frac{1}{x}\right)' = -\frac{1}{x^2}$$

$$\left(\frac{f}{g}\right)'(x_0) = f'(x_0)\cdot(h\circ g)(x_0) + (h\circ g)'(x_0)\cdot f(x_0)$$

$$= \frac{f'(x_0)}{g(x_0)} + h'(g(x_0))\cdot f(x_0)\cdot g'(x_0) = \frac{f'(x_0)}{g(x_0)} - \frac{g'(x_0)\cdot f(x_0)}{g^2(x_0)}$$

$$= \frac{f'(x_0)\cdot g(x_0) - g'(x_0) f(x_0)}{g^2(x_0)}$$

$$f(x) = \frac{(x+2)^2}{x} \qquad ((x+2)^2)' = 2(x+2)\cdot 1$$

$$f'(x) = \frac{2(x+2)\cdot x - 1\cdot(x+2)^2}{x^2} = \frac{2x^2+4x-x^2-4x-4}{x^2} = \frac{x^2-4}{x^2}$$

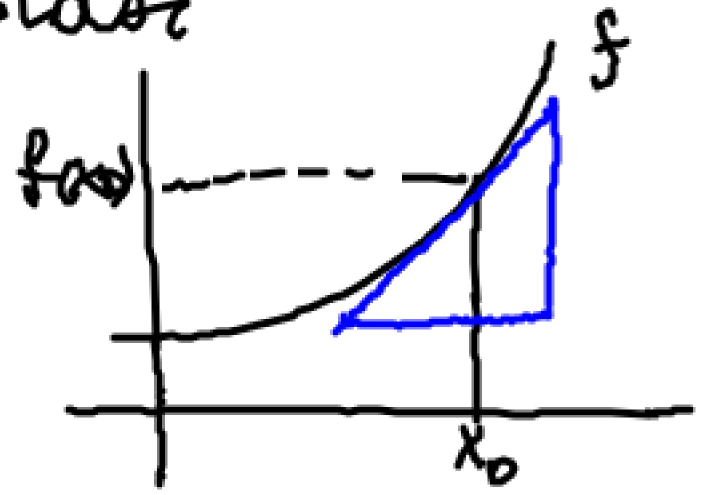
$$f(x) = \frac{(\cos(x)+2x)^2}{(5x+2)^3}$$

$$f'(x) = \frac{2(\cos(x)+2x)\cdot(2-\sin(x))(5x+2)^3 - 3(5x+2)^2\cdot 5\cdot(\cos x+2x)^2}{(5x+2)^6}$$

Věta 7.8 (Derivace inverzní funkce)

Nechť $f: J \to I$ je rostoucí nebo klesající $x_0 \in J$, $f^{-1}: I \to J$. Pokud existuje vlastní $f'(x_0) \neq 0$ potom existuje $(f^{-1})'(f(x_0))$ a platí

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$$



Důkaz:

$$1 = (id_J)'(x_0) = (f^{-1} \circ f)'(x_0) = (f^{-1})'(f(x_0)) \cdot f'(x_0)$$

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)} \iff \boxed{\underbrace{f}_{\ln}'(x_0) = \frac{1}{(\underbrace{f^{-1}}_{\exp})'(f(x_0))}}$$

$$\ln'(x) = f', \quad f^{-1} = \exp$$

$$\ln'(x) = \frac{1}{\exp'(\ln(x))} = \frac{1}{\exp(\ln x)} = \frac{1}{x}$$

$$f = \arcsin \qquad f^{-1} = \sin \qquad \cos x = \sqrt{1 - \sin^2 x}$$

$$\arcsin'(x) = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} =$$

$$= \frac{1}{\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\sqrt{1 - x^2}}$$

Diferenciál funkce

$f: J \to \mathbb{R}$, $x_0 \in J$, $A: \mathbb{R} \to \mathbb{R}$

diferenciál funkce f v bodě x_0 $\qquad df(x_0)$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-A(h)}{h} = 0$$

$$A: \mathbb{R} \to \mathbb{R}$$
$$A(h) = k\cdot h$$

Věta 7.10 $\exists\, df(x_0) \Leftrightarrow \exists\, f'(x_0) \qquad k = f'(x_0)$

Důkaz

$$0 = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-A(h)}{h} =$$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-k\cdot h}{h} = \underbrace{\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}}_{f'(x_0)} - k$$

$$f'(x_0) = k.$$

$f : J \longrightarrow \mathbb{R}$

$f' : A \subset J \longrightarrow \mathbb{R}$ A - interval

$(f')' : B \subset A \longrightarrow \mathbb{R}$ f''

$\vdots$

$$f^{(n)} = \left(f^{(n-1)}\right)'$$