Věta 6.14 1. Funkce exp je spojitá.

2. Pro každé $x, y \in \mathbb{R}$, platí $\exp(x+y) = \exp(x)\exp(y)$.

3. Funkce exp je rostoucí.

4. $\lim_{x \to \infty} \exp(x) = \infty$ a $\lim_{x \to -\infty} \exp(x) = 0$.

Důk.: 3. $\exp(x) > 0 \quad \forall x \in \mathbb{R}$

$$x < y$$

$$\frac{x^n}{n!} < \frac{y^n}{n!} \qquad \sum \frac{x^n}{n!} < \sum \frac{y^n}{n!}$$

4.

$$\exp(x) = e^x \qquad e > 1 \qquad \lim_{x \to \infty} e^x = \infty$$

$$\lim_{x \to -\infty} \exp(x) = \lim_{x \to +\infty} \exp(-x) = \frac{1}{\lim_{x \to \infty} \exp(x)} = 0$$

Důsledek 6.15.

1. Pro každé celé číslo k platí $\exp(k) = e^k$.
2. $\exp(\mathbb{R}) = (0, \infty)$.
3. Funkce exp je homeomorfismus.

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$
$$\exp(y) = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$$

$$\begin{array}{ccccc}
1 & x & \frac{x^2}{2!} & \frac{x^3}{3!} & \frac{x^4}{4!} \quad \ldots \\
y & xy & \frac{x^2 y}{2!} & \frac{x^3 y}{3!} & \ldots \\
\frac{y^2}{2!} & \frac{x y^2}{2!} & \frac{x^2 y^2}{2!\,2!} & \ldots & \\
\frac{y^3}{3!} & \frac{x y^3}{3!} & \ldots & & \\
\frac{y^4}{4!} & & & &
\end{array}$$

$$1 + (x+y) + \left(\frac{x^2}{2!} + xy + \frac{y^2}{2!}\right) + \ldots +$$
$$+ \left(\frac{x^n}{n!} + \frac{x^{n-1}\,y}{(n-1)!\,1!} + \frac{x^{n-2}\,y^2}{(n-2)!\,2!} + \ldots + \frac{x\,y^{n-1}}{1!\,(n-1)!} + \frac{y^n}{n!}\right) + \ldots$$

$$y_1 = \sum_{k=0}^{n} \frac{x^{n-k}\,y^k}{(n-k)!\,k!} = \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!}\, x^{n-k} y^k = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$
$$= \frac{(x+y)^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = \exp(x+y)$$

Logaritmus $\ln = \exp^{-1}$

Věta 6.16. 1. Funkce ln je spojitá.
2. Pro každé $x_1, x_2 \in (0,\infty)$ je $\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$.
3. Funkce ln je rostoucí.
4. $\lim_{x \to 0+} \ln(x) = -\infty$ a $\lim_{x\to\infty} \ln(x) = \infty$.

Exponenciální funkce o základu a.

$a > 0$ $\boxed{\exp_a(x) = e^{x \ln(a)}}$ $\exp_a : \mathbb{R} \to (0,\infty)$

$a \in (1,\infty)$

$\ln(a) > 0$

Věta 6.17 1. $\exp_a$ je spojitá.
2. $\forall x, y \in \mathbb{R}$ platí $\exp_a(x+y) = \exp_a(x)\exp_a(y)$.
3. Je-li $a \in (0,1)$ je $\exp_a$ klesající, pro $a=1$ konstantní a pro $a \in (1,\infty)$ rostoucí.
4. $a \in (0,1)$ $\lim_{x\to\infty} \exp_a(x) = 0$, $\lim_{x\to-\infty} \exp_a(x) = \infty$

$a \in (1,\infty)$ $\lim_{x\to\infty} \exp_a(x) = \infty$ $\lim_{x\to-\infty} \exp_a(x) = 0$

Goniometrické funkce

$\sin : \mathbb{R} \to [-1,1]$

$\cos : \mathbb{R} \to [-1,1]$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Věta 6.19. 1. Funkce sin, cos jsou spojité.

2. Funkce sin je lichá; funkce cos je sudá.

3. Pro každé $x, y \in \mathbb{R}$ platí

$$\sin(x+y) = \boxed{\sin(x)\cos(y)} + \boxed{\cos(x)\sin(y)}$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y).$$

$$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \_\_\_$$

$$\cos(e) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \_\_\_$$

$\sin x \cdot \cos y$

x	$-\frac{x^3}{3!}$	$\frac{x^5}{5!}$	$-\frac{x^7}{7!}$	$\dots$
$-\frac{xy^2}{2!}$	$\frac{x^3y^2}{3!\,2!}$	$-\frac{x^5y^2}{5!\,2!}$	$\dots$	
$\frac{xy^4}{4!}$	$-\frac{x^3y^4}{3!\,4!}$	$\dots$		
$-\frac{xy^6}{6!}$	$\dots$			

$\sin y \cos x$

y	$-\frac{y^3}{3!}$	$\frac{y^5}{5!}$	$-\frac{y^7}{7!}$	$\dots$
$-\frac{yx^2}{2!}$	$\frac{y^3x^2}{3!\,2!}$	$-\frac{y^5x^2}{5!\,2!}$	$\dots$	
$\frac{yx^4}{4!}$	$-\frac{y^3x^4}{3!\,4!}$	$\dots$		
$-\frac{yx^6}{6!}$	$\dots$			

$$(x+y) - \left(\frac{x^3}{3!} + \frac{yx^2}{2!} + \frac{y^2x}{2!} + \frac{y^3}{3!}\right) + \left(\frac{x^5}{5!} + \frac{x^4y}{4!\,1!} + \frac{x^3y^2}{3!\,2!} + \right.$$

$$\left. + \frac{x^2y^3}{2!\,3!} + \frac{xy^4}{1!\,4!} + \frac{y^5}{5!}\right) + \_\_\_ + \frac{1}{n!}\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} =$$

$$= \sin(x+y)$$

$$\pi = 4\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right)$$

Důsledek 6.20

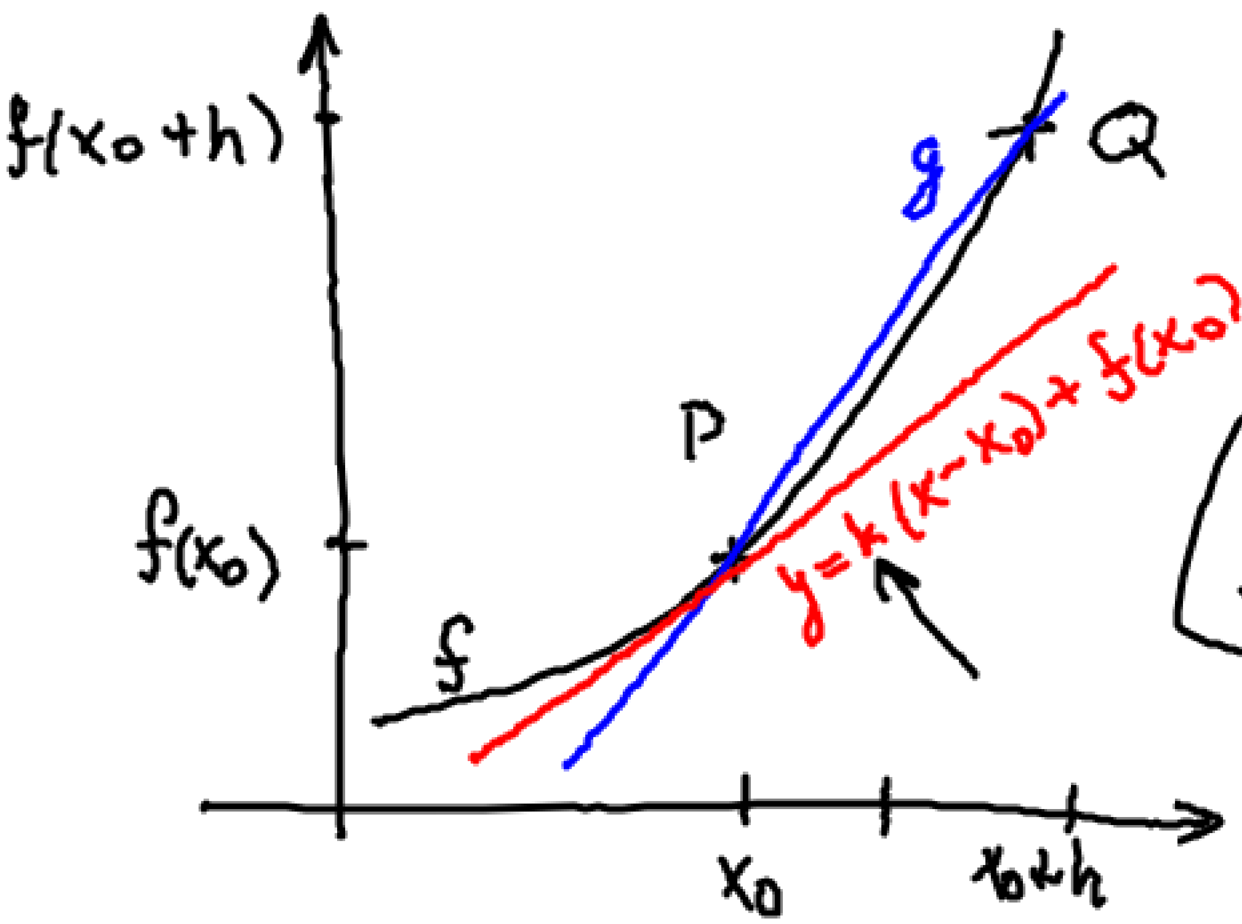
1. $\forall x \in \mathbb{R}$ $\sin(2x) = 2\sin(x)\cos(x)$, $\cos(2x) = \cos^2(x) - \sin^2(x)$
2. $\forall x \in \mathbb{R}$ $\sin^2(x) + \cos^2(x) = 1$.

Funkce tangens a kotangens.

Cyklometrické funkce.

DERIVACE

$f: J \longrightarrow \mathbb{R}$ $\quad$ $J \subset \mathbb{R}$ otevřený interval



P $(x_0, f(x_0))$

Q $(x_0+h, f(x_0+h))$

$$\boxed{g(x) = \frac{f(x_0+h)-f(x_0)}{h}(x-x_0)+f(x_0)}$$

směrnice $\boxed{\dfrac{f(x_0+h)-f(x_0)}{h}}$

$x \mapsto kx+q$

Derivace funkce f v bodě x_0:

$$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

$$f'_+(x_0)=\lim_{h\to 0+}\frac{f(x_0+h)-f(x_0)}{h}$$

$$f'_-(x_0)=\lim_{h\to 0-}$$

$X \subset J$ $\quad$ $f': X \longrightarrow \mathbb{R}$ $\quad$ $f'(x)=f'(x_0)$

$f: J \longrightarrow \mathbb{R} \quad f(x) = c \qquad f' = 0$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{c - c}{h} =$$

$$= \lim_{h \to 0} \frac{0}{h} = 0 \qquad (c)' = 0$$

$f: J \longrightarrow \mathbb{R} \quad f(x) = x^2 - 5x + 6$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h \to 0} \frac{(x_0+h)^2 - 5(x_0+h) + 6 - x_0^2 + 5x_0 - 6}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x_0^2} + 2hx_0 + h^2 \cancel{- 5x_0} - 5h \cancel{- x_0^2} \cancel{+ 5x_0}}{h} =$$

$$= \lim_{h \to 0} \frac{2hx_0 + h^2 - 5h}{h} = \lim_{h \to 0} 2x_0 + h - 5 =$$

$$= 2x_0 - 5$$

$pow_n \qquad pow_n' = n\, pow_{n-1} \qquad (x^n)' = n x^{n-1}$

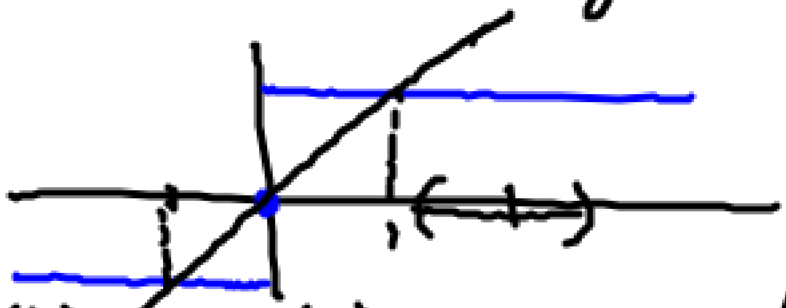
$$pow_n'(x_0) = \lim_{h \to 0} \frac{pow_n(x_0+h) - pow_n(x_0)}{h} = \lim_{h \to 0} \frac{(x_0+h)^n - x_0^n}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{x_0^n} + n x_0^{n-1} \cdot h + \binom{n}{2} x^{n-2} h^2 + \_\_ + n h^{n-1} x_0 + h^n \cancel{- x_0^n}}{h} =$$

$$= \lim_{h \to 0} \underbrace{n x_0^{n-1}} + \binom{n}{2} x^{n-2} \cdot h + \_\_ + n h^{n-2} x_0 + h^{n-1} = n x_0^{n-1}$$

Signum

$\mathrm{sgn}: \mathbb{R} \to \mathbb{R}$ $\qquad \mathrm{sgn}(x) = \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$



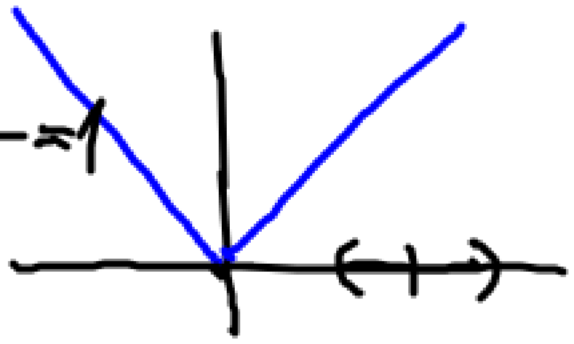
$$\mathrm{sgn}'_+(0) = \lim_{h\to 0+} \frac{\mathrm{sgn}(h)-\mathrm{sgn}(0)}{h} = \lim_{h\to 0+} \frac{1}{h} = \infty$$

$$\mathrm{sgn}'_-(0) = \lim_{h\to 0-} \frac{\mathrm{sgn}(h)-\mathrm{sgn}(0)}{h} = \lim_{h\to 0-} \frac{-1}{h} = \infty$$

Absolutní hodnota

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x<0 \end{cases}$$

$$\mathrm{abs}'_+(0) = \lim_{h\to 0+} \frac{|h|-|0|}{h} = \lim_{h\to 0+} \frac{h}{h} = 1$$



$$\mathrm{abs}'_-(0) = \lim_{h\to 0-} \frac{|h|-|0|}{h} = \lim_{h\to 0-} \frac{-h}{h} = -1$$

Věta 7.2 Nechť $f: J \to \mathbb{R}$ má v $x_0 \in J$ konečnou derivaci, potom f je v x_0 spojitá.

Důkaz $\lim\limits_{x \to x_0} f(x) = f(x_0)$

$$\lim_{x \to x_0} \left(f(x) - f(x_0)\right) = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}(x - x_0)\right) =$$

$$= \underbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{f'(x_0)} \cdot \underbrace{\lim_{x \to x_0} x - x_0}_{0} = 0$$

$f(x) = \sqrt[3]{x}$

$$f'(0) = \lim_{h \to 0} \frac{\sqrt[3]{h} - 0}{h} =$$

$$= \lim_{h \to 0} h^{-2/3} = \lim_{h \to 0} \frac{1}{\sqrt[3]{h^2}} = \underline{\infty}$$

Věta 7.3 Buď (f_n) posloupnost funkcí $f_n : J \to \mathbb{R}$, $x_0 \in J$. Pokud $\sum f_n$ konverguje stejnoměrně na$_{k f}$ J. a $\forall n\ \exists f_n'(x_0) = a_n$.
Pak řada $\sum a_n$ konverguje $f'(x_0)$ existuje a platí

$$f'(x_0) = \sum a_n, \quad f'(x_0) = \sum f_n'(x_0)$$

Příklad: $\exp(x)$

$$\exp'(x) = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right)' = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \dots$$

$$= \exp(x) \qquad (e^x)' = e^x.$$