$\Omega_X \times G \to \Omega_W \times \Omega_X$. Let α denote the natural map from Ω_X to $\Omega_Y = \Omega_X/G$ and let α' be the corresponding map from $\Omega_W \times \Omega_X$ to $\Omega_W \times \Omega_Y$. The set $\alpha'(\Delta)$ is a T-invariant closed subset of $\Omega_X \times \Omega_Y$ and $\pi_W(\alpha'(\Delta)) = \Omega_W$, $\pi_Y(\alpha'(\Delta)) = \alpha(\Omega_X) = \Omega_Y$. Since $Y \perp W$, we have $\alpha'(\Delta) = \Omega_W \times \Omega_Y$. From this it follows that for every $(\omega,\xi) \in \Omega_W \times \Omega_X$ there exists $g \in G$ with $(\omega,\xi g) \in \Delta$. This means that the product ΔG coincides with $\Omega_W \times \Omega_X$.

We shall show now that Δ itself coincides with $\Omega_W \times \Omega_X$, and this will complete the proof. Let V be a closed set in G with non-empty interior and consider the product ΔV. Since G is compact there exist $g_1, \cdots, g_r \in G$ with $\bigcup_{i=1}^{r} Vg_i = G$. Hence $\bigcup_{i=1}^{r} \Delta V g_i = \Delta G = \Omega_W \times \Omega_X$. The sets $\Delta V g_i$ being closed, it follows that one of them, and therefore each of them, has non-empty interior. But $\Delta V g_i$ is T-invariant since Δ is T-invariant and T commutes with the action of G. However, we also know that $W \times X$ is ergodic by Proposition II.11; hence $\Delta V g_i = \Omega_W \times \Omega_X$ and so $\Delta V = \Omega_W \times \Omega_X$. Now we can find a decreasing sequence of closed sets $\{V_n\}$ each with non-empty interior and with $\cap V_n = \{\text{identity}\}$. Since $\Delta V_n = \Omega_W \times \Omega_X$ we conclude that $\Delta = \Omega_W \times \Omega_X$. This proves the theorem.

We conclude with the statement of two open problems.

Problem F: Is it true that $\mathscr{W} \times \mathscr{E} \subset \mathscr{E}$?

Problem G: Describe the classes $\mathscr{M}^\perp$ and $\mathscr{D}^\perp$.

Part III. Properties of Minimal Sets.

1. Minimal Subsets of Groups. Let G be a compact abelian group and T an endomorphism of Ω such that $X = (G,T)$ is an $\mathscr{F}$-flow. In Part II, §3 (D), we showed that this is the case if G is a finite-dimensional torus, and T an endomorphism of G corresponding to an integer matrix with no roots of unity among its eigenvalues. It will also be the case if $G = \Lambda \times \Lambda \times \Lambda \times \cdots$, where Λ is a compact abelian group and T represents the shift operator.

With this assumption, $X \perp \mathscr{M}$ by Theorem II.2, and in particular, X is disjoint from its own minimal subflows. Using this, we shall be able to show that the minimal sets in G are "small" in a certain sense.

Definition III.1. *Let G be a topological group and T an endomorphism of G. A closed T-invariant subset A of G is said to be* restricted *if $AB = G$ for some closed T-invariant set $B \subset G$, implies $B = G$.*

Here AB is the collection of all products $\alpha\beta$, $\alpha \in A$, $\beta \in B$. Note that the notion of a restricted subset depends on the endomorphism T and the group structure of G.

Definition III.2. *A* basis *of a group G is a set B with the property that each element in G is a product of finitely many elements in B.*

LEMMA III.1. *Let G be a non-trivial compact metrizable group, and suppose that (G,T) is ergodic. Then a restricted set cannot be a basis of G.*

Proof: If A is a basis then $\cup A^n = G$. Then some A^n contains an open set (we assume A is closed) and since A is T-invariant, so is A^n. By ergodicity, $A^n = G$. But then $A^{n-1} = G$, $A^{n-2} = G, \cdots, A = G$, $\{e\} = G$, and so G is trivial if A is restricted.

The main result of this section is

THEOREM III.1. *If (G,T) is an $\mathcal{F}$-flow, then every minimal set for (G,T) is restricted.*

The proof of this theorem is based on the following lemma. This lemma is reminiscent of the familiar fact that if a number divides a product and is relatively prime to one of the factors, it necessarily divides the remaining factor.

LEMMA III.2. *Let X and Y be disjoint flows, and suppose there is a flow Z and a homomorphism $X \times Z \xrightarrow{\pi} Y$. Then for each $\xi \in \Omega_X$, the map $\zeta \to \pi(\xi,\zeta)$ takes Ω_Z onto Ω_Y.*

Proof: Let $W = X \times Z$. We have $W \xrightarrow{\pi_X^W} X$, $W \xrightarrow{\pi_Z^W} Z$, and $W \xrightarrow{\pi} Y$. Since $X \perp Y$, there must exist a homomorphism $W \xrightarrow{\gamma} X \times Y$ such that $\pi_X^W = \pi_X\gamma$, and $\pi = \pi_Y\gamma$. Then $\gamma(\xi,\zeta) = (\xi,\pi(\xi,\zeta))$. Now γ takes Ω_W onto $\Omega_X \times \Omega_Y$, so for each (ξ,η) there is a point $\omega \in \Omega_W$ with $\gamma(\omega) = (\xi,\eta)$. But this means that the equation $\eta = \pi(\xi,\zeta)$ has a solution for each pair (ξ,η).

Proof of Theorem III.1. Let M be a minimal set of the $\mathcal{F}$-flow (G,T). If M is not restricted, we may find a closed T-invariant set $B \subset G$ with $MB = G$ and $B \neq G$. Let $X = (M,T)$, $Y = (G,T)$, and $Z = (B,T)$. We have a map $\pi{:}M \times B \to G$ defined by $\pi(\mu,\beta) = \mu\beta$ which takes $M \times B$ onto G. Hence π defines a homomorphism of $X \times Z$ to Y. But $X \perp Y$ since X is minimal and Y is an $\mathcal{F}$-flow. By Lemma III.2, $\pi(\mu,B) = G$, or $\mu B = G$ for any $\mu \in M$. But G is a group, and this implies that $B = G$.

2. Relative Dimension, Hausdorff Dimension and Topological Entropy. Let $\Lambda = \{0,1,\cdots, a-1\}$, where a is a positive integer, set $\Omega = \Lambda \times \Lambda \times \Lambda \times \cdots$, and let T denote the shift transformation of Ω. Λ can be given a group structure in several ways; for each of these, Ω becomes a group and T an endomorphism of Ω. For each of these structures the results of the last section apply and a minimal set for (Ω,T) will be "restricted" with respect to the various group structures on Ω. The question arises whether it is possible to introduce a quantitative notion of the size of sets for which all minimal sets will be "small". There are several notions which suggest themselves, for all of which it is true that a "small" set is a restricted set. We shall, however, see that in spite of evidence to the contrary, minimal sets need not be small in any of these senses.

There is a correspondence between Ω and the unit interval which is almost one-one. Namely, we associate with the sequence $(\omega_1,\omega_2,\omega_3,\cdots)$ the real number $\sum_1^\infty \omega_n a^{-n}$. The operator T corresponds to the operator τ_a on $[0,1]$ which is defined by $\tau_a x = ax$ modulo 1. τ_a becomes continuous if we identify 0 and 1, i.e., if we take as its domain the additive group of reals modulo one. Calling this group K we obtain a flow (K,τ_a). For a set $A \subset \Omega$ we shall denote by A^* the corresponding set in K. A T-invariant set A in Ω is minimal for (Ω,T) if and only if A^* is a minimal set for (K,τ_a). In determining the "size" of a T-invariant set $A \subset \Omega$ we may consider interchangeably properties of A and properties of A^*.

One such measure of the size of a set in Ω or in K is given by its *Hausdorff dimension.* This exists for every subset B of K and we denote it by $D(B)$. Then $0 \leq D(B) \leq 1$. There is a related notion which is more useful for certain purposes and which we refer to as the *relative dimension.* Partition K into N equal intervals: $K = \bigcup_{0}^{N-1} [m/N,(m+1)/N]$, and let $\nu(B,N)$ be the number of these that contain points of B. If

$$\lim_{N\to\infty} \frac{\log \nu(B,N)}{\log N} \tag{1}$$

exists, we call it the *relative dimension* of B, and denote it by $d(B)$. It is easy to see that whenever $d(B)$ exists, it satisfies $d(B) \geq D(B)$. One establishes readily the relationship

$$\frac{\nu(B,N_2)}{\nu(B,N_1)} < 3 + \frac{N_2}{N_1},$$

which shows that if the limit (1) exists for a subsequence $\{N_k\}$ satisfying $N_{k+1}/N_k < M < \infty$, then the limit exists. In particular, it suffices to establish the existence of $\lim_{n\to\infty} \frac{\log \nu(B,g^n)}{n}$ for some integer g.

Finally, there is a notion of size which is applicable to subsets of Ω. Let $A \subset \Omega$ and denote by $\mu(A,n)$ the number of Λ-valued n-tuples $(a_1,a_2,\cdots,a_n)$ which coincide with $(\omega_1,\omega_2,\cdots,\omega_n)$ for some point $\omega = (\omega_1,\omega_2,\cdots,\omega_n,\cdots) \in A$. If

$$\lim_{n\to\infty} \frac{\log \mu(A,n)}{n} \tag{2}$$

exists, it is called the *topological entropy* of A and denoted $\mathcal{E}(A)$. If A is T-invariant, then (A,T) is a flow. In this case $\mathcal{E}(A)$ always exists and it may be seen that $\mathcal{E}(A)$ coincides with the *topological entropy* of the flow (A,T) as defined in [1].

PROPOSITION III.1. *Let A be a T-invariant subset of Ω and let A^* denote the corresponding subset of K. Then $d(A^*)$ and $\mathcal{E}(A)$ exist and*

$$D(A^*) = d(A^*) = \frac{\mathcal{E}(A)}{\log a} \tag{3}$$

Remark: From this proposition it follows that the Hausdorff dimension of the classical Cantor set is log 2/log 3. For, the Cantor set corresponds to the set of sequences in Ω $(\Lambda = \{0,1,2\})$ in which 1 does not appear. This clearly has topological entropy log 2.

Proof: By the T-invariance of A, it is clear that $\mu(A,n+m) \leq \mu(A,n) \times \mu(A,m)$. It is well known that this implies the existence of the limit in (2); hence $\mathscr{E}(A)$ exists. Consider now $\nu(A^*,a^n)$. The interval $[m/a^n,(m+1)/a^n]$ will contain a point of A^* if and only if the a-adic expansion of m, $m = \sum b_i a^{i-1}$, is such that $(b_1,b_2,\cdots,b_n)$ coincides with the initial n-block of a sequence in A. Hence $\nu(A^*,a^n) = \mu(A,n)$. Thus

$$\lim_{n\to\infty} \frac{\nu(A^*,a^n)}{n}$$

exists and coincides with $\mathscr{E}(A)$. By our remarks concerning relative dimension, we conclude that $d(A^*)$ exists and $d(A^*) = \mathscr{E}(A)/\log a$.

We must still show that the Hausdorff dimension of A^* coincides with $d(A^*)$. Since one always has $d(A^*) \geq D(A^*)$, we must show that $D(A^*) \geq d(A^*)$. Recall that $D(A^*) \geq \delta$ if, for every covering of A^* by intervals I, of sufficiently small length, $\sum |I_j|^\delta > c > 0$. We claim now that in applying this criterion, it suffices to consider coverings with intervals of the form $I_j = [m_j/a^{n_j},(m_j+1)/a^{n_j}]$. For, as one can see, it is possible to replace an arbitrary covering by one with intervals of this sort, thereby multiplying $\sum |I_j|^\delta$ by no more than $2a^\delta$. It will suffice therefore to show that if

$$A^* \subset \bigcup_{j=1}^{J} [m_j/a^{n_j},(m_j+1)/a^{n_j}]$$

and $\delta < d(A^*)$, then $\sum_1^J a^{-\delta n_j} \geq 1$.

We can restate this in terms of the set A. We denote by R' the collection of all n-tuples of elements of Λ, $R' = \bigcup_1^\infty \Lambda^n$. R' is a semigroup if we multiply by juxtaposition: $(a_1,\cdots,a_n)(b_1,\cdots,b_m) = (a_1,\cdots,a_n,b_1,\cdots,b_m)$. We denote by R the subset of R' consisting of n-tuples which occur as blocks in sequences of A. Thus $\mu(A,n)$ is the number of elements of R of length n. Notice that $\rho_1\rho_2 \in R$ implies that both ρ_1 and ρ_2 belong to R. We shall say that ρ' is *divisible* by ρ if $\rho' = \rho\rho_1$ for some $\rho_1 \in R'$. Also $l(\rho)$ will denote the length of ρ. With these preliminaries we may restate what must be proved as follows. If $\{\rho_i\}$ is a finite collection of elements of R such that each $\rho \in R$ of sufficiently great length is divisible by some ρ_i, and if $\delta < d(A^*)$, then $\sum a^{-\delta l(\rho_i)} \geq 1$.

Suppose to the contrary that $\sum a^{-\delta l(\rho_i)} < 1$. Then

$$\sum a^{-\delta l(\rho_{i_1}\rho_{i_2}\cdots\rho_{i_n})} < \infty,$$

where the sum is taken over the semigroup generated in R' by $\{\rho_i\}$. We now claim that there is a finite set of elements $\{\rho'_j\}$ such that every $\rho \in R$ can be expressed as a product $\rho = \rho_{i_1}\rho_{i_2}\cdots\rho_{i_n}\rho'_j$, for some sequence $\rho_{i_1},\cdots,\rho_{i_n}$ in $\{\rho_i\}$ and for some ρ'_j. The reason is that each $\rho \in R$ is divisible by some ρ_i, $\rho = \rho_{i_1}\rho'$ with $\rho' \in R$, provided $l(\rho)$ is large enough. But from this it follows that

$$\sum_{R} a^{-\delta l(p)} < \infty .$$

In other words $\sum \mu(A,n)a^{-\delta n} < \infty$. However if $\delta < d(A^*) = \mathscr{E}(A)/\log a$, then for n sufficiently large, $\mu(A,n) > a^{\delta n}$ and the series in question must diverge. This proves the proposition.

3. Deterministic Flows. In [1] the analogy between "entropy" and "topological entropy" is developed. For a number of flows it may be shown that the flow X supports a measure μ such that the entropy of the process (X,μ) equals the topological entropy of X. In any case the former never exceeds the latter. The topological entropy of a product is the sum of the topological entropies of the factors, and the topological entropy of a flow is at least as great as that of any of its factors. This analogy suggests defining a class of "deterministic" flows by the condition that the topological entropy vanish. A deterministic flow has the property that any process supported by it is deterministic in the sense of Part I, §3. It may be that the converse is true as well.

Let A and B denote τ_a-invariant subsets of K, and let $A + B$ denote their sum, i.e., the set of all sums $\alpha + \beta$, $\alpha \in A$, $\alpha \in B$. Evidently, $\nu(A+B,N) \leqslant \nu(A,2N)\nu(B,2N)$. From this we find that $d(A+B) \leqslant d(A) + d(B)$.

PROPOSITION III.2. *If A is a τ_a-invariant closed subset of K such that (A,τ_a) is deterministic, then A is restricted.*

Proof: If (A,τ_a) is deterministic, then $d(A) = 0$. Hence $A + B = K$ implies $d(B) = 1$. Let B correspond to a T-invariant subset $B' \subset \Omega$. $d(B) = 1$ implies $\mathscr{E}(B') = \log a$. Now if B' is a proper subset of Ω, there is some n-tuple, for some n, of elements of Λ which does not occur in the sequences of B'. But then $\mu(B',n) \leqslant a^n - 1$ for some n, and so $\mu(B',nm) \leqslant (a^n - 1)^m$, whence $\mathscr{E}(B') \leqslant \log (a^n - 1)/n \log a < 1$. This proves the proposition.

There appears to be a certain amount of evidence to support the conjecture that minimal flows are deterministic. For one thing, a subset A of K is restricted if the flow (A,τ_a) is either minimal or deterministic. Secondly the class $\mathscr{B}^{\perp}$ of flows disjoint from Bernoulli flows coincides with $\mathscr{M}$, whereas the processes disjoint from Bernoulli processes are precisely the deterministic processes. Finally, the most familiar examples of minimal sets do correspond to deterministic flows. For instance, the recurrent sequences described by Robbins [12] lead to deterministic flows, and the minimal set described by Gottschalk [7] is deterministic. We shall see however that this evidence is misleading and that there exist minimal flows which are not deterministic, or, equivalently, minimal sets for (K,τ_a) with positive Hausdorff dimension.

THEOREM III.2. *There exist minimal sets with positive topological entropy.*

Proof: According to Proposition II.6, the flow defined by a recurrent sequence (see Definition II.3) is minimal. Moreover, if $\{\xi(n)\}$ is a Λ-valued sequence, and X is the flow it defines, it is possible to compute the topological entropy of X directly from $\{\xi(n)\}$. Namely, if H_n is the number of distinct n-tuples of symbols of the form $(\xi(l+1), \cdots, \xi(l+n))$ for some $l \geq 0$, then $\mu(A,n) = H_n$, A being the subset of Ω defined by the flow X. To construct a minimal flow with positive entropy we shall construct a recurrent sequence $\{\xi(n)\}$ for which H_n grows exponentially with n.

Decompose the natural numbers $J = \{1,2,3, \cdots\}$ into a disjoint union of arithmetic progressions: $J = \bigcup_1^\infty J_r$, $J_r = c_r + d_r J$, $J_r \cap J_s = \varnothing$ for $r \neq s$. For each r choose a number $a(r) \in \Lambda$, and define $\{\xi(n)\}$ by setting $\xi(n) = a(r)$ if $n \in J_r$. It is easily seen that the sequence $\{\xi(n)\}$ is recurrent. In fact, each m-tuple $(\xi(1), \xi(2), \cdots, \xi(m))$ recurs periodically.

The condition $J = \bigcup_1^\infty J_r$ implies that $\sum 1/d_r \leq 1$. We would expect equality, but one readily sees that the d_r may be chosen to increase as rapidly as we like. Moreover, it is not hard to see that the set of initial points $\{c_r\}$ may be made as dense as we like in J, simply by choosing $\{d_r\}$ to be increasing sufficiently rapidly. In particular, we may choose $\{J_r\}$ such that

$$\limsup c_r/r < 2 . \tag{4}$$

For such m there will be infinitely many r such that $c_{r+m} - c_{r+1} < 2m$. Denote this set of integers r by Q_m.

Now let $\{y_n\}$ denote a sequence of independent identically distributed random variables, each having for its value the symbols in Λ, all occurring with positive probability. For any choice of m symbols, $a_1, a_2, \cdots, a_m$, the probability is 0 that for each $r \in Q_m$, $(y_{r+1}, y_{r+2}, \cdots, y_{r+m}) \neq (a_1, a_2, \cdots, a_m)$. It follows that there exists a sample sequence $\{y_n\} = \{\eta_n\}$ such that for every m and every m-tuple $(a_1, a_2, \cdots, a_m)$, there is an $r \in Q_m$ with $(\eta_{r+1}, \eta_{r+2}, \cdots, \eta_{r+m}) = (a_1, a_2, \cdots, a_m)$.

With this sequence $\{\eta_n\}$ we form a recurrent sequence $\{\xi(n)\}$, setting $\xi(n) = \eta_r$ for $n \in J_r$. We find that, for every m-tuple $(a_1, a_2, \cdots, a_m)$, an r exists with $c_{r+m} - c_{r+1} < 2m$, and with $\xi(c_{r+j}) = a_j, j = 1, \cdots, m$. This means that there is a value of n such that in the sequence $\xi(n+1), \cdots, \xi(n+2m)$, the values $a_1, \cdots, a_m$ occur somewhere and in that order, though not necessarily consecutively (namely, take $n = c_{r+1}$). Now each fixed block $\xi(n+1), \cdots, \xi(n+2m)$ can give rise to at most $\binom{2m}{m} < 4^m$ m-tuples $(a_1, a_2, \cdots, a_n)$. It follows that the number H_{2m} for our sequence $\{\xi(n)\}$ exceeds $(a/4)^m$. As a result, if $a > 4$, the flow associated with $\{\xi(n)\}$ will have positive topological entropy. This proves our theorem.

We remark that this theorem has also been proven independently and in sharper form by F. Hahn and Y. Katznelson (as yet unpublished). They show that the flow X may be chosen to be strictly ergodic, and such that if μ is the unique invariant probability measure supported by X, then the entropy of the process (X, μ) is positive.

4. $\mathscr{M} \times \mathscr{M} \not\subset \mathscr{D}$. The class of deterministic flows introduced in the last section has the property that products of flows in the class are again in the class and factors of flows in the class are in it. The object of the remainder of this part is to construct a class with the same properties and containing the class of minimal flows. By Theorem III.2, the class of deterministic flows does not suffice for this purpose. The problem may also be described in another way. If we define a *deterministic sequence* as one which is defined for a deterministic flow, then it is easily seen that continuous functions of finitely many deterministic sequences $[\zeta(n) = f(\xi_1(n), \xi_2(n), \cdots, \xi_r(n))]$ are again deterministic sequences. This is a direct consequence of the closure properties of the class of deterministic flows. Since neither $\mathscr{M}$ nor $\mathscr{S}$ is closed under multiplication, there will exist functions of recurrent sequences which are not recurrent. The question arises, what properties are shared by sequences formed in this way?

The fact that products of minimal flows need not even be semi-simple, or equivalently, that a composite sequence $\{\xi_1(n), \xi_2(n)\}$ whose components are recurrent need not be recurrent, may be seen as follows. As before let Ω be the product $\Lambda \times \Lambda \times \Lambda \times \cdots$, with Λ a finite set.

PROPOSITION III.3. *Let $X = (A,T)$ be a subflow of (Ω,T) such that A is infinite. Then $X \times X$ is not semi-simple.*

Remark: In particular, if X is minimal and A is not finite, the hypotheses are fulfilled and $X \times X$ is a product of minimal flows which is not semi-simple. This proposition also shows that a flow of finite type cannot be distal unless it is periodic. For we recall from Part II, §3 (G), that X is distal if and only if $X \times X$ is semi-simple.

Proof: On Ω, and therefore on A, are defined coordinate functions x_n: $x_n(\omega_1, \omega_2, \cdots) = \omega_n \in \Lambda$. Suppose $X \times X$ were semi-simple. Then for any continuous function φ on $A \times A$, the sequence $\varphi(T^n\omega, T^n\omega')$ would be recurrent (Proposition II.6). Using this we shall show that on A, x_1 is a continuous function of $x_2, x_3, x_4, \cdots$. To show this it suffices to show that if $\omega_1, \omega_2 \in A$ and if $x_n(\omega_1) = x_n(\omega_2)$ for $n = 2,3,4, \cdots$, then $x_1(\omega_1) = x_1(\omega_2)$. But if $X \times X$ is semi-simple, then $x_n(\omega_1) - x_n(\omega_2)$ represents a recurrent sequence. If it vanishes for $n = 2,3,4, \cdots$, it must vanish for $n = 1$. Thus we may write $x_1 = F(x_2, x_3, x_4, \cdots)$ with F a continuous function on A. Now it is easily seen that since each x_i takes on only finitely many variables, $x_1 = F(x_2, x_3, \cdots, x_m)$. Since A is T-invariant, we have $x_n = F(x_{n+1}, x_{n+2}, \cdots, x_{n+m-1})$. However, there exist only finitely many sequences satisfying these conditions, and hence A must be finite. This proves the proposition.

An explicit example of a minimal set of finite type is

$$A = \text{set of sequences } \{\xi(n) = \text{sgn sin } (n\alpha + \theta)\}$$

where α is fixed and θ varies between 0 and 2π. When $n\alpha + \theta = \nu\pi$, ν an integer, we define sgn sin $(n\alpha + \theta)$ as either $+1$ or -1. That is, θ of the form $\nu\pi - k\alpha$ gives rise to two sequences $\{\xi(n)\}$, both of which are included in A. We leave it to the reader to verify that (A,T) is minimal. That the flow is not distal can be seen directly by considering limits of translates of $\{\xi^{\pm}(n)\}$ where

$$\xi^{\pm}(n) = \begin{cases} \text{sgn sin } n\alpha & n \neq 0 \\ \pm 1 & n = 0 \, . \end{cases}$$

The above example shows directly that sums and products of recurrent sequence are, in general, not recurrent. For $\xi^+(n) - \xi^-(n)$ and $\xi^+(n)\xi^-(n)$ are evidently not recurrent.

5. Binary Sequence Spaces. Our object in the next two sections is to construct a class of flows of finite type (Part II, §3 (A)), closed under products and passage to factor flows (that is, if X and Y are of finite type, $X \xrightarrow{\alpha} Y$ and X is in the class, then Y should also be in the class) and containing all minimal flows of finite type. It is clear that what is sought is a relatively restricted class of flows; the more restricted this class, the more information is to be had concerning products of minimal flows.

As yet we have only been successful in dealing with flows of finite type, but in all likelihood a similar situation will present itself in the general case.

In the present section, all the flows to be considered are subflows of (Ω_2, T), where Ω_2 is the space of all $\{0,1\}$-valued sequences $\xi = (\xi(1), \xi(2), \cdots, \xi(n), \cdots)$ and T is the shift transformation. It will be convenient to regard Ω_2 as a ring, addition and multiplication of sequences being carried out term by term, treating $\{0,1\}$ as the field with two elements. During the remainder of this section, the expression "invariant set" will refer to a closed, T-invariant subset of Ω_2. A minimal subset Ω_2 will also be understood to be an invariant set which is minimal for (Ω_2, T).

By Definition III.1, a subset A of Ω_2 is restricted (with respect to the additive structure of Ω_2) if $A + B = \Omega_2$ implies $B = \Omega_2$. Here A and B are invariant sets. It is clear that the sums of two restricted sets is restricted. We do not know if the product of restricted sets is restricted. Nevertheless, we have the following proposition which is a sharpening of Theorem III.1:

PROPOSITION III.4. *If A is a restricted subset of Ω_2 and M is a minimal set in Ω_2, then MA is restricted.*

Proof: Suppose B is an invariant set satisfying $MA + B = \Omega_2$. We show that $B = \Omega_2$. Consider the flows (M,T), (A,T), (B,T), and (Ω_2,T). By hypothesis, there is a map π of $M \times A \times B$ onto Ω_2 which defines a homomorphism of the product of the first three flows onto the last; namely $\pi(\mu,\alpha,\beta) = \mu\alpha + \beta$. Recalling that (M,T), being minimal, is disjoint from (Ω_2,T) which is an $\mathscr{F}$-flow, we are in a position to apply Lemma III.2. We conclude that for each $\mu \in M$, $\pi(\mu,A,B) = \Omega_2$, or, $\mu A + B = \Omega_2$. Fix μ and choose $\omega' \in \Omega_2$ in the form $\omega' = 1 - \mu + \mu\omega$ with ω an unspecified element of Ω_2. For each choice of ω there must exist $\alpha \in A$, $\beta \in B$ such that

(5) $$\mu\alpha + \beta = 1 - \mu + \mu\omega \, .$$

Multiply both sides by $(1-\mu)(1-\beta)$ and recall that for $\xi \in \Omega_2$, $\xi^2 = \xi$, or $\xi(1-\xi) = 0$. We thus have

$$(1-\mu)(1-\beta) = 0,\ (1-\beta) = \mu(1-\beta)\ . \tag{6}$$

Now multiply both sides of (5) by $(1-\beta)$:

$$(1-\beta)\mu\alpha = (1-\beta)\mu\omega$$

and since $(1-\beta)\mu = 1-\beta$,

$$(1-\beta)\alpha = (1-\beta)\omega$$

and

$$\omega = \alpha + \beta(\omega - \alpha)\ . \tag{7}$$

(7) shows that every element in Ω_2 is a sum of an element of A and a multiple of an element of B:

$$\Omega_2 = A + B\Omega_2\ .$$

But $B\Omega_2$ is an invariant set and A_1 by hypothesis is restricted. We conclude that $B\Omega_2 = \Omega_2$. Now in Ω_2 the unit element 1 has a unique representation as a product: $1 = 1 \cdot 1$. It follows that $1 \in B$. We have shown that $MA + B = \Omega_2$ implies that $1 \in B$. But in addition, $MA + (B + C) = \Omega_2$ for any invariant set C. It follows that $1 \in B + C$. Take $C = \{1\}$; we conclude that $0 \in B$. Hence $0 \in B + C$ for any invariant set C. This means every invariant set C intersects $-B$. Thus B itself must intersect every invariant set, and hence it must contain every minimal set in Ω_2. So B contains every periodic sequence in Ω_2, and since these are dense, we infer that $B = \Omega_2$. This proves the proposition.

By repeated application of this proposition and the fact that sums of restricted sets are restricted, we conclude that any set in Ω_2 of the form

$$A = \sum_{i=1}^{m} M_{i1}M_{i2} \cdots M_{il_i} \tag{8}$$

is restricted, the M_{ij} denoting minimal sets.

Definition III.3. *A sequence $\xi \in \Omega_2$ is* restricted *if it belongs to a restricted set. We denote by R the family of all restricted sequences.*

R is a T-invariant subset of Ω_2, but it is not closed. However if $\xi \in R$, then all limits of translates of ξ are in R. It is also clear that $R \neq \Omega_2$. To see this, observe that (Ω_2, T), being ergodic, has dense orbits. Hence there are sequences $\omega \in \Omega_2$ belonging to no invariant subsets other than Ω_2 itself. These points (which in fact constitute a residual set in Ω_2) are not restricted.

PROPOSITION III.5. *An invariant set $A \subset \Omega_2$ is restricted if and only if $A \subset R$.*

Proof: The necessity is clear, so we turn to the sufficiency. Suppose $A \subset R$ and $A + B = R$, A and B being invariant sets. As remarked above there exists $\omega \in \Omega_2$ whose orbit is dense in Ω_2. Write $\omega = \alpha + \beta$, $\alpha \in A$,

$\beta \in B$. Now $\alpha \in R$, hence $\alpha \in A'$, where A' is a restricted set. He[nce] $\omega \in A' + B$. The latter set is, however, invariant; hence $\Omega_2 = A' + B$ and so $B = \Omega_2$. This proves the proposition.

THEOREM III.3. *Let R_0 denote the subring of Ω_2 generated by all recurrent binary sequences. Then $R_0 \subset R$.*

Proof: Every element in R_0 is of form

$$\xi = \sum_{i=1}^{m} \xi_{i1}\xi_{i2} \cdots \xi_{il_i},$$

where the ξ_{ij} are recurrent sequences. Each ξ_{ij} belongs to a minimal set M_{ij} and hence ξ belongs to a set of the form (8). It follows that ξ is restricted.

R_0 is T-invariant and not closed. It also has the property that a limit of translates of an element of R_0 is again in R_0. Thus R_0 is a union of closed T-invariant sets.

COROLLARY. *R_0 is a set of the first category in Ω_2. R itself may not be a ring. By Proposition III.5, it is an additive subgroup. Also by Proposition III.4, it is a module over the ring R_0.*

6. R_0-Flows. The ring R_0 of binary sequences may be identified with a ring of subsets of the natural numbers. The latter ring we denote R^*_0. We shall make use of R^*_0 in studying general finite-valued sequences. Let Λ be a finite set and suppose $\{\xi(n)\}$ is a Λ-valued sequence.

Definition III.4. *A sequence ξ is* R_0-measurable *if the subsets of the form $\{n:\xi(n) \in \Delta\}$, Δ a subset of Λ, belong to R^*_0.*

THEOREM III.4. *(a) If a sequence is R_0-measurable, then so are all translates and limits of translates of the sequence; (b) Any function of finitely many R_0-measurable sequences $(\xi(n) = f(\xi_1(n), \cdots, \xi_m(n)))$ is R_0-measurable; (c) Recurrent sequences are R_0-measurable.*

Proof: (a) and (b) follow by virtue of the fact that R^*_0 is a ring and R_0 has the invariance properties in question. For (c), suppose that $\{\xi(n)\}$ is recurrent. If we let $\eta(n) = 1$ or 0 according as $\xi(n) \in \Delta$ or not, then $\{\eta(n)\}$ is a function of $\{\xi(n)\}$ and hence itself recurrent. Hence $\eta \in R_0$ and $\{\xi(n)\}$ is R_0-measurable.

Finally we introduce the flows that correspond to R_0-measurable sequences.

Definition III.5. *Let X be a subflow of (Ω,T), where $\Omega = \Lambda \times \Lambda \times \Lambda \times \cdots$, and Λ is finite. If each sequence of Ω_X is R_0-measurable, then X is said to be an* R_0-flow. *The class of R_0-flows will be denoted by $\mathcal{R}_0$.*

THEOREM III.5. *(a) All semi-simple flows of finite type are in $\mathcal{R}_0$; (b) $\mathcal{R}_0 \times \mathcal{R}_0 \subset \mathcal{R}_0$; (c) Subflows of flows in $\mathcal{R}_0$ are in $\mathcal{R}_0$; (d) Factors of flows in $\mathcal{R}_0$ are in $\mathcal{R}_0$ provided they are of finite type; (e) Non-trivial Bernoulli flows do not occur in $\mathcal{R}_0$.*

Proof: (a), (b) and (c) are immediate from Theorem III.4. To prove (d) assume that $X \in \mathscr{R}_0$ and that $X \xrightarrow{\varphi} Y$, where Y is a flow of finite type. Writing the points of Ω_X and Ω_Y as sequences, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots) \in \Omega_X$, $\beta = (\beta_1, \beta_2, \beta_3, \cdots) \in \Omega_Y$, we have

$$\beta_n = \Phi(\alpha_n, \alpha_{n+1}, \cdots)$$

for some continuous function Φ on Ω_X. Since both the α_n and the β_m are finite-valued, Φ can depend only on finitely many coordinates:

$$\beta_n = \Phi(\alpha_n, \alpha_{n+1}, \cdots, \alpha_{n+r}) .$$

But then if α is R_0-measurable, so is β, and so $Y \in \mathscr{R}_0$.

Finally (e) follows from the fact that R_0 is a proper subset of Ω_2. Now the space of a Bernoulli flow consists of all Λ-sequences for some space Λ, and the only ring with respect to which all these sequences are measurable is the ring of all subsets of the natural numbers. This proves the theorem.

7. A Necessary Condition for an R_0-flow. Let A be an invariant subset of $\Omega = \Lambda \times \Lambda \times \Lambda \times \cdots$, where Λ is a finite set. Let Q be an infinite subset of the natural numbers. We say A is *free on* Q if for any choice of numbers $q_1, \cdots, q_r \in Q$, and $\lambda_1, \cdots, \lambda_r \in \Lambda$, we can find a sequence $\alpha \in A$ with $\alpha_{q_1} = \lambda_1, \cdots, \alpha_{q_r} = \lambda_r$.

THEOREM III.6. *If* $(A,T) \in \mathscr{R}_0$ *then* A *cannot be free on a relatively dense set (see Definition II.3).*

Proof: If A is free on Q and if $A' = \{\alpha' : \alpha'_n = \varphi(\alpha_n), \alpha \in A\}$ for some function $\varphi: \Lambda \to \Lambda'$, then A' is again free on Q. Choose φ as a two-valued function so that $A' \subset \Omega_2$. If $(A,T) \in \mathscr{R}_0$ then $(A',T) \in \mathscr{R}_0$ and so $A' \subset R_0$. We wish to show that if A' is free on Q, then Q cannot be relatively dense.

If $\beta \in \Omega_2$, then the set of n for which $\beta(n) = 1$ will be called the *support* of β. With A' as before, let B denote the set of all $\beta \in \Omega_2$ such that A' is free on the support of β. B is T-invariant, and because the notion of being free depended on conditions regarding finitely many coordinates at one time, B is seen to be closed. Suppose some $\beta \in B$ has for its support a relatively dense set. Clearly all translates of β have the same property (uniformly) and so do limits of these. The set $\overline{\{T^n\beta\}}$ then contains a minimal set M whose members have relatively dense support. In particular M does not degenerate to $\{0\}$. Let $\mu \in M$. Since A' is free on the support of μ, we can write $A'\mu = \Omega_2\mu$, or $\Omega_2 = A' + (1-\mu)\Omega_2$. If we set $M' = \{1-\mu : \mu \in M\}$, then *a fortiori*, $\Omega_2 = A'M'\Omega_2$. Since A' is a restricted set, $M'\Omega_2 = \Omega_2$. Hence $1 \in M'$ and so $0 \in M$. But M is minimal and we assumed $M \neq \{0\}$. Hence no $\beta \in B$ has relatively dense support, and this proves the theorem.

Combining Theorems III.5 and III.6 we find that, beginning with semi-simple flows of finite type and forming subflows, factor flows of finite type and product flows we always obtain flows satisfying the conditions of the theorem. Namely, the space of the flow is a sequence space which cannot be free on a relatively dense set.

We note that deterministic flows also satisfy the conditions of Theorem III.6. For if a set A is free on a relatively dense set, it is clear that $\mu(A,n)$ increases exponentially with n.

We conclude this part by formulating two problems which we have not been able to solve:

Problem H. Is $R \subset \Omega_2$ a ring, and if so, does it coincide with R_0?

Problem I. Is the condition in Theorem III.6 sufficient as well as necessary for an R_0-flow?

Part IV. A Problem in Diophantine Approximation.

1. Minimal Sets on Tori. We let K^r denote the r-dimensional torus considered as an additive group: $K^r = \mathbf{R}^r/\mathbf{Z}^r$. The endomorphisms of K^r correspond to $r \times r$ integer matrices, and these form a semigroup $E(r)$. Each $\sigma \in E(r)$ determines a flow on K^r, and under certain mild conditions, (K^r,σ) will be an $\mathscr{F}$-flow. Theorem III.1 then gives us certain information about minimal sets of (K^r,σ): If M is minimal and B is an invariant set, then $M + B = K^r$ implies $B = K^r$.

In all this we have been dealing exclusively with a "one-parameter" semigroup of transformations. It is not difficult to see that the various notions we have used generalize to the situation where a more complicated semigroup acts on the space. Namely, we can allow flows (Ω,Σ) where Σ is now not a single transformation, but an abelian semigroup of transformations. The notions of minimal flows, semi-simple flows, $\mathscr{F}$-flows, and the notion of disjointness all generalize to this situation. Developing the analogy we may obtain a generalization of Theorem III.1 to abelian groups on which an abelian semigroup of endomorphisms acts. Specifically, we may obtain the following result whose usefulness will develop presently.

PROPOSITION IV.1. *Let Σ denote a commutative semigroup of endomorphisms of the r-torus K^r. We assume (i) that the adjoint semigroup of endomorphisms Σ^* of the dual group $\mathbf{Z}^r$ possesses no finite invariant subset in $\mathbf{Z}^r - \{0\}$, and (ii) that there exists a prime q with the property that all* det σ, $\sigma \in \Sigma$, *are relatively prime to q. Then if M and B are two closed Σ-invariant subsets of K^r and M is minimal with respect to these properties, $M + B = K^r$ implies $M = K^r$.*

Remark: We do not know whether condition (ii) is essential. Condition (i) is indispensable because, for example, if Σ were finite, the conclusion of the proposition would be false.

In proving this proposition, we shall, in fact, not proceed in the manner indicated, that is, we shall not redevelop the machinery alluded to in the case of flows in the wider sense. Since the results of this section may have some independent interest we have chosen to present the proof in a disguised form, avoiding the language of flows.

Proof of Proposition IV.1. Let $\Sigma^{(n)}$ denote the subsemigroup of $\sigma \in \Sigma$ for which $\sigma \equiv 1$ modulo q^n. Each $\sigma \in \Sigma$ has some power lying in $\Sigma^{(n)}$. For with respect to the finite ring $\mathbf{Z}/q^n\mathbf{Z}$, det σ is a unit for $\sigma \in \Sigma$. Hence, modulo