Příklady strana 200

1) $f: X \to \mathbb{R}$ $\qquad X = \mathbb{R} \qquad (X, \mathcal{L}, \lambda)$

$x \mapsto c \qquad c > 0$

$X \in \mathcal{L}$ je měřitelná

$\lambda(\mathbb{R}) = \infty$

$\int f \, d\lambda = c \cdot \mu(\mathbb{R}) = \infty \quad \Rightarrow$ není integrovatelná

2) $f, g \in JNMF$ – jednoduché, nezáporné měřitelné fce.

• $\int f \, d\mu \neq \infty$

$$\int f+g \, d\mu = \underbrace{\int f \, d\mu}_{\neq \infty} + \int g \, d\mu \neq \infty$$

• $\int f \, d\mu < \infty, \quad \int g \, d\mu < \infty$

$\Rightarrow \int f+g \, d\mu = \int f \, d\mu + \int g \, d\mu < \infty$

3) f, g nezáporné, jednoduché, integrovatelné

$$(f \vee g)(x) = \max(f(x), g(x)) = \frac{1}{2}\left(f(x) + g(x) + |f(x) - g(x)|\right)$$

$$(f \wedge g)(x) = \min(f(x), g(x)) = \frac{1}{2}\left(f(x) + g(x) - |f(x) - g(x)|\right)$$

$\Rightarrow f \vee g, \; f \wedge g$ nezáporné jednoduché měřitelné (Věta 4.3.3)

f, g integrovatelné $\Rightarrow f \vee g, \; f \wedge g$ integrovatelné

jedna z nich není int. $\Rightarrow f \vee g$ není integrovatelná

$f \wedge g$ je integrovatelná

f, g nejsou integrovatelné $\Rightarrow f \wedge g, f \vee g$ není integrovatelné

4) $f \vee g$ integrovatelná $\Rightarrow f, g$ jsou integrovatelné

5) $$f(x) = \begin{cases} 1 & x \in \langle 0, \infty) \\ 0 & x \in (-\infty, 0) \end{cases} \qquad g(x) = \begin{cases} 1 & x \in (-\infty, 0) \\ 0 & x \in \langle 0, \infty) \end{cases}$$

$\Rightarrow (f \wedge g)(x) = 0 \quad$ pro $\forall x \in \mathbb{R}$

$\int f \, d\mu = \infty \qquad \int g \, d\mu = \infty$

$\int f \wedge g \, d\mu = \underline{0} < \infty$

6) $X = \mathbb{N}$ $\quad \mathcal{S} = 2^X$

$\mu(A) = \# A$

$f: X \to \mathbb{R}$ integrovatelná

Charakterizace f: jen konečně mnoho prvků je zobrazí jinam než do 0

$$\int f \, d\mu = c_1 \mu(E_1) + \ldots + c_n \mu(E_n) < \infty$$

$$\iff c_i \cdot \mu(E_i) < \infty \iff \mu(E_i) < \infty$$

7) $X = \mathbb{N}$, $\quad \mathcal{S} = 2^X$

$\mu(A) = \# A$

$$f(x) = \begin{cases} 2 & x \text{ je sudá} \\ 0 & x \text{ je lichá} \end{cases} \qquad f(x) = \begin{cases} 5 & x > 5 \\ 0 & \text{jinak} \end{cases}$$

8) a) $\int 2 \chi_{\mathbb{Q}} \, d\lambda = 2 \cdot \mu(\mathbb{Q}) = 2 \cdot 0 = 0 < \infty \Rightarrow$ integrovatelná

b) $\int 3 \chi_E + 2 \chi_{\langle 0,1 \rangle} \, d\lambda = 3 \mu(E) + 2 \cdot \mu(\langle 0,1 \rangle) = 3 \cdot 2 + 2 \cdot 1 = 2 < \infty \Rightarrow$ integrovatelná

c) $G \subset \langle 0,1 \rangle$ je lebesguovsky měřitelná

$$\int 4 \chi_G + 2 \chi_{\langle -1,1 \rangle} \, d\lambda = 4 \cdot \mu(G) + 2 \cdot \mu(\langle -1, 1 \rangle) \leq$$

$$\leq 4 \cdot \mu(\langle 0,1 \rangle) + 2 \cdot \mu(\langle -1,1 \rangle) = 4 \cdot 1 + 2 \cdot 2 = 8 < \infty$$

$\Rightarrow$ integrovatelná

Příklady strana 210

1) $(X, \mathcal{S}, \mu)$

$X = \mathbb{N} \qquad \mathcal{S} = 2^X \qquad \mu(E) = \# E$

$f \equiv 0 \qquad \Rightarrow \int f \, d\mu = 0$

$f(k) = \begin{cases} 1 & k \leq 5 \\ 0 & k > 5 \end{cases} \qquad \int f \, d\mu = 5$

Obecně: $f = \sum_{i=1}^{\infty} c_i \, \chi_{\{i\}}$

$$\int f \, d\mu = \int \sum c_i \, \chi_{\{i\}} \, d\mu = \sum \int c_i \, \chi_{\{i\}} \, d\mu = \sum c_i < \infty$$

$\Rightarrow$ posloupnost $\{c_i\}$ musíme volit tak, aby řada $\sum c_i$ konvergovala

2) a) $f(k) = \begin{cases} 1 & k \leq 5 \\ 0 & k > 5 \end{cases}$

$$\int f \, d\mu = \sum_{k=1}^{\infty} \mu(\{i\}) \cdot c_i = 5$$

b) $f(k) = \frac{1}{k} \qquad \Rightarrow \int f \, d\mu = \infty$

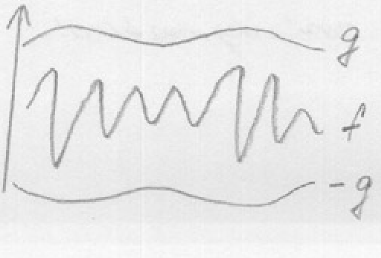
c) $f(k) = -\frac{1}{k} \qquad \Rightarrow \int f \, d\mu = -\infty$

d) $f(k) = \begin{cases} 1 & k \text{ je sudé} \\ -1 & k \text{ je liché} \end{cases}$

$$\Rightarrow \int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu = 1 \cdot \infty - 1 \cdot \infty = \infty - \infty$$

- NEVÍME
- NEMÁ INTEGRÁL

3) f měřitelná, g integrovatelná, $|f| \leq g$
je f integrovatelná?



$$-g \leq f \leq g$$

(z věty 5.2.9)

$$\underbrace{-\int g \, d\mu}_{> -\infty} \leq \int f \, d\mu \leq \underbrace{\int g \, d\mu}_{< \infty}$$

4) f integrovatelná $X \to \mathbb{R}_0 = [-\infty, \infty]$

Dokažte, že existuje integrovatelná $g: X \to \mathbb{R}$ a $f = g$ s.v.

$E = \{x : |f(x)| = \infty\}$ má míru 0 (Věta 5.2.13)

Definujme
$$g(x) = \begin{cases} f(x) & x \notin E \\ 1 & x \in E \end{cases}$$

$$\int g \, d\mu = \int_{(X \setminus E) \cup E} g \, d\mu = \int_{X \setminus E} g \, d\mu + \int_E g \, d\mu = \int_{X \setminus E} f \, d\mu + 1 \cdot \mu(E) =$$

$$= \int_{X \setminus E} f \, d\mu < \int f \, d\mu < \infty$$

5) f, g integrovatelné

$\int f \vee g \, d\mu$, $\int f \wedge g \, d\mu$?

$$\int f \vee g \, d\mu = \int \frac{1}{2}(f + g + |f - g|) \, d\mu = \frac{1}{2} \underbrace{\int f \, d\mu}_{< \infty} + \frac{1}{2} \underbrace{\int g \, d\mu}_{< \infty} + \frac{1}{2} \int |f - g| \, d\mu < \infty$$

$$\left| \int |f - g| \, d\mu \right| < \infty \overset{\text{věta 5.2.11.}}{\iff} \left| \int f - g \, d\mu \right| < \infty \iff \left| \underbrace{\int f \, d\mu}_{< \infty} - \underbrace{\int g \, d\mu}_{< \infty} \right| < \infty$$

$$\int f \wedge g \, d\mu = \int \frac{1}{2}(f + g - |f - g|) \, d\mu = \frac{1}{2} \int f \, d\mu + \frac{1}{2} \int g \, d\mu - \frac{1}{2} \int |f - g| \, d\mu < \infty.$$

6) $(X, \mathcal{S})$ $X = (0,1)$ $\mathcal{S} = \{X, \emptyset\}$ - σ-algebra

$\mu(\emptyset) = 0$

$\mu(X) = 1$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap (0,1) \\ -1 & x \notin \mathbb{Q} \cap (0,1) \end{cases}$$

$$\int f \, d\mu = \int_{X \setminus \mathbb{Q} \cup (\mathbb{Q} \cap X)} f \, d\mu = \int_{X \setminus \mathbb{Q}} f \, d\mu + \int_{\mathbb{Q} \cap X} f \, d\mu = \int f \, \chi_{X \setminus \mathbb{Q}} \, d\mu + \int f \, \chi_{\mathbb{Q} \cap X} \, d\mu$$

$$= 1 \cdot \underbrace{\mu(X \cap \mathbb{Q})}_{= ?} + (-1) \cdot \underbrace{\mu(X \setminus \mathbb{Q})}_{= ?} \implies \text{není integrovatelná}$$

$|f|(x) = 1$ $\quad \int |f| \, d\mu = 1 \cdot \mu(X) = \underline{\underline{1}}$