Limita

Buď f funkce, $x_0 \in \mathbb{R}$, $a \in \mathbb{R}$. Říkáme, že funkce f má v bodě x_0 limitu a, jestliže ke každému $\varepsilon > 0$ existuje $\delta > 0$ tak, že pro všechna $x \neq x_0$, $x \in (x_0 - \delta, x_0 + \delta)$ je $f(x) \in (a - \varepsilon, a + \varepsilon)$.

Ekvivalentně:

$$\lim_{x \to x_0} f(x) = a \iff \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in \mathbb{R}: \quad 0 < |x - x_0| < \delta \Rightarrow |f(x) - a| < \varepsilon.$$

Limita zprava: $\lim_{x \to x_0^+} f(x) = a \iff \forall \varepsilon > 0 \ \exists \delta > 0: \quad \forall x \in (x_0, x_0 + \delta) \Rightarrow |f(x) - a| < \varepsilon.$

Limita zleva: $\lim_{x \to x_0^-} f(x) = a \iff \forall \varepsilon > 0 \ \exists \delta > 0: \quad \forall x \in (x_0 - \delta, x_0) \Rightarrow |f(x) - a| < \varepsilon.$

Vlastnosti:

- $\forall x \in \mathbb{R}: f(x) = c \Rightarrow \lim_{x \to a} f(x) = c$ pro $\forall a \in \mathbb{R}$
- $\lim_{x \to a} (f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $\lim_{x \to a} (c \cdot f(x)) = c \cdot \lim_{x \to a} f(x)$
- $\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} (f(x)) \cdot \lim_{x \to a} g(x)$
- $\lim_{x \to a} \frac{f(x)}{h(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} h(x)}$

(Ovšem za předpokladu, že $\lim_{x \to a} f(x)$, $\lim_{x \to a} g(x)$, $\lim_{x \to a} h(x)$ existují a navíc $\lim_{x \to a} h(x) \neq 0$)

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = a \iff \lim_{x \to x_0} f(x) = a$$

PŘÍKLADY:

- $\lim\limits_{x\to 0} \dfrac{x^2-1}{2x^2-x-1} = \dfrac{0^2-1}{2\cdot 0^2-0-1} = \dfrac{-1}{-1} = \underline{\underline{1}}$

- $\lim\limits_{x\to 2} \dfrac{x^2+5}{x^2-3} = \dfrac{2^2+5}{2^2-3} = \dfrac{9}{1} = \underline{\underline{9}}$

- $\lim\limits_{x\to 1} \dfrac{x^2-1}{2x^2-x-1} = \lim\limits_{x\to 1} \dfrac{(x-1)(x+1)}{(x-1)(2x+1)} = \lim\limits_{x\to 1} \dfrac{x+1}{2x+1} = \underline{\underline{\dfrac{2}{3}}}$

- $\lim\limits_{x\to 4} \dfrac{x^2-8x+16}{x^2-7x+12} = \lim\limits_{x\to 4} \dfrac{(x-4)(x-4)}{(x-4)(x-3)} = \lim\limits_{x\to 4} \dfrac{x-4}{x-3} = \underline{\underline{0}}$

- $\lim\limits_{x\to 2} \dfrac{x^2-5x+6}{x^2-7x+10} = \lim\limits_{x\to 2} \dfrac{(x-3)(x-2)}{(x-2)(x-5)} = \lim\limits_{x\to 2} \dfrac{x-3}{x-5} = \underline{\underline{\dfrac{1}{3}}}$

- $\lim\limits_{x\to 2} \dfrac{x^2+x-6}{x^2-x-2} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+3)}{(x-2)(x+1)} = \lim\limits_{x\to 2} \dfrac{x+3}{x+1} = \underline{\underline{\dfrac{5}{3}}}$

- $\lim\limits_{x\to a} \dfrac{x^2-a^2}{x-a} = \lim\limits_{x\to a} \dfrac{(x-a)(x+a)}{(x-a)} = \lim\limits_{x\to a} (x+a) = \underline{\underline{2a}}$

- $\lim\limits_{x\to 3} \dfrac{x^2-5x+6}{x^2-8x+15} = \lim\limits_{x\to 3} \dfrac{(x-3)(x-2)}{(x-3)(x-5)} = \lim\limits_{x\to 3} \dfrac{x-2}{x-5} = \dfrac{1}{-2} = \underline{\underline{-\dfrac{1}{2}}}$

- $\lim\limits_{x\to 1} \dfrac{x^3-3x+2}{x^4-4x+3} = \lim\limits_{x\to 1} \dfrac{(x-1)(x^2+x-2)}{(x-1)(x^3+x^2+x-3)} = \lim\limits_{x\to 1} \dfrac{x^2+x-2}{x^3+x^2+x-3} =$

  $= \lim\limits_{x\to 1} \dfrac{(x-1)(x+2)}{(x-1)(x^2+2x+3)} = \dfrac{3}{6} = \underline{\underline{\dfrac{1}{2}}}$

- $\lim\limits_{x\to 1} \dfrac{x^4-3x+2}{x^5-4x+3} = \lim\limits_{x\to 1} \dfrac{(x-1)(x^3+x^2+x-2)}{(x-1)(x^4+x^3+x^2+x-3)} =$

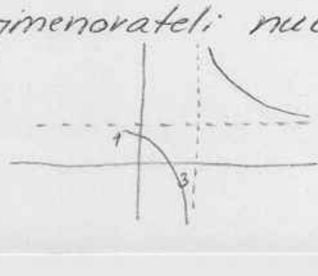
  $= \lim\limits_{x\to 1} \dfrac{x^3+x^2+x-2}{x^4+x^3+x^2+x-3} = \underline{\underline{1}}$

- $\lim\limits_{x\to 2} \dfrac{x^3-2x^2-4x+8}{x^4-8x^2+16} = \lim\limits_{x\to 2} \dfrac{(x-2)(x^2-4)}{(x-2)(x^3+2x^2-4x-8)} =$

  $= \lim\limits_{x\to 2} \dfrac{x^2-4}{x^3+2x^2-4x-8} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)(x^2+4x+4)} = \dfrac{4}{16} = \underline{\underline{\dfrac{1}{4}}}$

Výraz upravujeme dokud je ve jmenovateli nula.

POZOR: $\lim\limits_{x\to 3} \dfrac{x-2}{x-3}$ neexistuje.



$$\bullet \lim_{x\to 0} \frac{(1+x)(1+2x)(1+3x)-1}{x} = \lim_{x\to 0} \frac{6x^3+6x^2+2x^2+3x^2+6x+1-1}{x} =$$

$$= \lim_{x\to 0} \frac{6x^3+11x^2+6x}{x} = \lim_{x\to 0} (6x^2+11x+6) = \underline{6}$$

$$\bullet \lim_{x\to 0} \frac{(1+x)^5-(1+5x)}{x^2+5x} = \lim_{x\to 0} \frac{(1+5x+10x^2+10x^3+5x^4+x^5)-(1+5x)}{x^2+5x}$$

$$= \lim_{x\to 0} \frac{10x^2+10x^3+5x^4+x^5}{x^2+5x} = \lim_{x\to 0} \frac{x(10x+10x^2+5x^3+x^4)}{x(x+5)} =$$

$$= \lim_{x\to 0} \frac{10x+10x^2+5x^3+x^4}{x+5} = \underline{\underline{0}}$$

$$\bullet \lim_{x\to 0} \frac{(1+x)^5-(1+5x)}{x^2+x^5} = \lim_{x\to 0} \frac{10x^2+10x^3+5x^4+x^5}{x^2+x^5} =$$

$$= \lim_{x\to 0} \frac{x^2(10+10x+5x^2+x^3)}{x^2(1+x^3)} = \lim_{x\to 0} \frac{10+10x+5x^2+x^3}{1+x^3} = \underline{\underline{10}}$$

$$\bullet \lim_{x\to 2} \frac{(x^2-x-2)^{30}}{(x^3-12x+16)^{10}} = \lim_{x\to 2} \frac{((x+1)(x-2))^{20}}{((x-2)(x^2+2x-8))^{10}} =$$

$$= \lim_{x\to 2} \frac{(x-2)^{20}(x+1)^{20}}{(x-2)^{10}(x^2+2x-8)^{10}} = \lim_{x\to 2} \frac{(x-2)^{10}(x+1)^{20}}{((x-2)(x+4))^{10}} =$$

$$= \lim_{x\to 2} \frac{(x-2)^{10}(x+1)^{20}}{(x-2)^{10}(x+4)^{10}} = \lim_{x\to 2} \frac{(x+1)^{20}}{(x+4)^{10}} = \frac{3^{20}}{6^{10}} = \frac{3^{20}}{3^{10}\cdot 2^{10}} =$$

$$= \frac{3^{10}}{2^{10}} = \underline{\underline{\left(\frac{3}{2}\right)^{10}}}$$

$$\bullet \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \underline{\underline{\frac{1}{2}}}$$

$$\bullet \lim_{x\to -1} \frac{x+1}{\sqrt{x+5}-2} = \lim_{x\to -1} \frac{(x+1)(\sqrt{x+5}+2)}{(\sqrt{x+5}-2)(\sqrt{x+5}+2)} =$$

$$= \lim_{x\to -1} \frac{(x+1)(\sqrt{x+5}+2)}{x+5-4} = \lim_{x\to -1} \frac{(x+1)(\sqrt{x+5}+2)}{x+1} =$$

$$= \lim_{x\to -1} (\sqrt{x+5}+2) = \underline{\underline{4}}$$

$$\bullet \lim_{x \to \cdot 3} \frac{x+3}{\sqrt{x+4}-1} = \lim_{x \to -3} \frac{(x+3)(\sqrt{x+4}+1)}{(\sqrt{x+4}-1)(\sqrt{x+4}+1)} = \lim_{x \to -3} \frac{(x+3)(\sqrt{x+4}+1)}{x+4-1}$$

$$= \lim_{x \to -3} \frac{(x+3)(\sqrt{x+4}+1)}{x+3} = \lim_{x \to -3} (\sqrt{x+4}+1) = \underline{\underline{2}}$$

$$\bullet \lim_{x \to 7} \frac{2-\sqrt{x-3}}{x^2-49} = \lim_{x \to 7} \frac{(2-\sqrt{x-3})(2+\sqrt{x-3})}{(x^2-49)(2+\sqrt{x-3})} = \lim_{x \to 7} \frac{4-(x-3)}{(x^2-49)(2+\sqrt{x-3})} =$$

$$= \lim_{x \to 7} \frac{7-x}{(x-7)(x+7)(2+\sqrt{3+x})} = \lim_{x \to 7} \frac{-(x-7)}{(x-7)(x+7)(2+\sqrt{x-3})} =$$

$$= \lim_{x \to 7} \frac{-1}{(x+7)(2+\sqrt{x-3})} = -\frac{1}{14 \cdot 4} = -\underline{\underline{\frac{1}{56}}}$$

$$\bullet \lim_{x \to 5} \frac{\sqrt{x-1}-2}{x-5} = \lim_{x \to 5} \frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{(x-5)(\sqrt{x-1}+2)} = \lim_{x \to 5} \frac{x-1-4}{(x-5)(\sqrt{x-1}+2)} =$$

$$= \lim_{x \to 5} \frac{1}{\sqrt{x-1}+2} = \underline{\underline{\frac{1}{4}}}$$

$$\bullet \lim_{x \to 1} \frac{\sqrt{x+3}-2}{x-1} = \lim_{x \to 1} \frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)} = \lim_{x \to 1} \frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} =$$

$$= \lim_{x \to 1} \frac{x-1}{(x-1)(\sqrt{x+3}+2)} = \lim_{x \to 1} \frac{1}{\sqrt{x+3}+2} = \underline{\underline{\frac{1}{4}}}$$

$$\bullet \lim_{x \to 3} \frac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9} = \lim_{x \to 3} \frac{(\sqrt{x+13}-2\cdot\sqrt{x+1})(\sqrt{x+13}+2\cdot\sqrt{x+1})}{(x+3)(x-3)(\sqrt{x+13}+2\cdot\sqrt{x+1})} =$$

$$= \lim_{x \to 3} \frac{x+13-4(x+1)}{(x+3)(x-3)(\sqrt{x+13}+2\cdot\sqrt{x+1})} = \lim_{x \to 3} \frac{-3x+9}{(x+3)(x-3)(\sqrt{x+13}+2\sqrt{x+1})}$$

$$= \lim_{x \to 3} \frac{-3(x-3)}{(x-3)(x+3)(\sqrt{x+13}+2\sqrt{x+1})} = \lim_{x \to 3} \frac{-3}{(x+3)(\sqrt{x+13}+2\sqrt{x+1})}$$

$$= -\frac{3}{6\cdot(4+4)} = -\frac{3}{48} = -\underline{\underline{\frac{1}{16}}}$$

$$\bullet \lim_{x \to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2} = \lim_{x \to 4} \frac{(\sqrt{1+2x}-3)(\sqrt{1+2x}+3)(\sqrt{x}+2)}{(\sqrt{x}-2)(\sqrt{1+2x}+3)(\sqrt{x}+2)} =$$

$$= \lim_{x \to 4} \frac{(1+2x-9)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)} = \lim_{x \to 4} \frac{2(x-4)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)} =$$

$$= \lim_{x \to 4} \frac{2(\sqrt{x}+2)}{\sqrt{1+2x}+3} = \frac{8}{6} = \underline{\underline{\frac{4}{3}}}$$

• $\lim\limits_{x\to\infty} \frac{x^2-1}{2x^2-x-1} = \lim\limits_{x\to\infty} \frac{x^2(1-\frac{1}{x^2})}{x^2(2-\frac{1}{x}-\frac{1}{x^2})} = \lim\limits_{x\to\infty} \frac{1-\frac{1}{x^2}}{2-\frac{1}{x}-\frac{1}{x^2}} = \frac{1-0}{2-0-0} = \underline{\underline{\frac{1}{2}}}$

• $\lim\limits_{x\to\infty} \frac{(2x-3)^{20}(3x+2)^{30}}{(2x-1)^{50}} = \lim\limits_{x\to\infty} \frac{(x(2-\frac{3}{x}))^{20}\cdot(x(3+\frac{2}{x}))^{30}}{(x(2-\frac{1}{x}))^{50}} =$

$= \lim\limits_{x\to\infty} \frac{x^{20}(2-\frac{3}{x})^{20}\cdot x^{30}\cdot(3+\frac{2}{x})^{30}}{x^{50}\cdot(2-\frac{1}{x})^{50}} = \lim\limits_{x\to\infty} \frac{(2-\frac{3}{x})^{20}\cdot(3+\frac{2}{x})^{30}}{(2-\frac{1}{x})^{50}} =$

$= \frac{2^{20}\cdot 3^{30}}{2^{50}} = \frac{3^{30}}{2^{30}} = \underline{\underline{\left(\frac{3}{2}\right)^{30}}}$

Důležité limity: $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

$\lim\limits_{x\to 0} \frac{e^x-1}{x} = 1$

$\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$

$\lim\limits_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

Využití těchto limit:

• $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \lim\limits_{x\to 0} \frac{(1-\cos x)\cdot(1+\cos x)}{x^2\cdot(1+\cos x)} = \lim\limits_{x\to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} =$

$= \lim\limits_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \lim\limits_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \lim\limits_{x\to 0} \frac{1}{1+\cos x} =$

$= \left(\lim\limits_{x\to 0} \frac{\sin x}{x}\right)^2 \cdot \lim\limits_{x\to 0} \frac{1}{1+\cos x} = 1^2 \cdot \frac{1}{1+1} = \underline{\underline{\frac{1}{2}}}$

• $\lim\limits_{x\to\infty} \left(\frac{2x+5}{2x+3}\right)^{x+1} = \lim\limits_{x\to\infty} \left(\frac{2x+3+2}{2x+3}\right)^{x+1} = \lim\limits_{x\to\infty} \left(1+\frac{2}{2x+3}\right)^{x+1} =$

$= \lim\limits_{x\to\infty} \left(1+\frac{\frac{1}{2}\cdot 2}{\frac{1}{2}\cdot(2x+3)}\right)^{x+1} = \lim\limits_{x\to\infty} \left(1+\frac{1}{x+\frac{3}{2}}\right)^{x+1} =$

$= \lim\limits_{x\to\infty} \left(1+\frac{1}{x+\frac{3}{2}}\right)^{x+3/2-1/2} = \underbrace{\lim\limits_{x\to\infty} \left(1+\frac{1}{x+\frac{3}{2}}\right)^{x+3/2}}_{=e} \cdot \left(1+\frac{1}{x+\frac{3}{2}}\right)^{-1/2} =$

$= e\cdot\left(1+\frac{1}{\infty}\right)^{-1/2} = e\cdot 1 = \underline{e}$

$a^2 - b^2 = (a-b)(a+b)$
$a^2 + b^2 = (a-b)^2 + 2ab$
$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$

Nevlastní limita funkce:

$\lim_{x \to a} f(x) = +\infty \iff \forall K \ \exists \delta > 0 : 0 < |x-a| < \delta \Rightarrow f(x) > K$

$\lim_{x \to a} f(x) = -\infty \iff \forall K \ \exists \delta > 0 : 0 < |x-a| < \delta \Rightarrow f(x) < K$

Nevlastní limity funkce zleva a zprava v bodě a:

$\lim_{x \to a^+} f(x) = +\infty \iff \forall K \ \exists \delta > 0 \ \forall x \in (a, a+\delta) \Rightarrow f(x) > K$

$\lim_{x \to a^+} f(x) = -\infty \iff \forall K \ \exists \delta > 0 \ \forall x \in (a, a+\delta) \Rightarrow f(x) < K$

$\lim_{x \to a^-} f(x) = +\infty \iff \forall K \ \exists \delta > 0 \ \forall x \in (a-\delta, a) \Rightarrow f(x) > K$

$\lim_{x \to a^-} f(x) = -\infty \iff \forall K \ \exists \delta > 0 \ \forall x \in (a-\delta, a) \Rightarrow f(x) < K$

Limita funkce v nevlastním bodě:

$\lim_{x \to \infty} f(x) = A \iff \forall \varepsilon > 0 \ \exists c \ \forall x > c : |f(x) - A| < \varepsilon$

$\lim_{x \to -\infty} f(x) = A \iff \forall \varepsilon > 0 \ \exists c \ \forall x < c : |f(x) - A| < \varepsilon.$

Nevlastní limita funkce v nevlastním bodě:

$\lim_{x \to \infty} f(x) = +\infty \iff \forall K \ \exists c \ \forall x > c : f(x) > K$

$\lim_{x \to -\infty} f(x) = +\infty \iff \forall K \ \exists c \ \forall x < c : f(x) > K$

$\lim_{x \to \infty} f(x) = -\infty \iff \forall K \ \exists c \ \forall x > c : f(x) < K$

$\lim_{x \to -\infty} f(x) = -\infty \iff \forall K \ \exists c \ \forall x < c : f(x) < K.$

Spojitost funkce

Funkce f se nazývá spojitá v bodě $x_0 \in \mathbb{R}$, platí-li $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Funkce f se nazývá zprava (zleva) spojitá v bodě x_0, platí-li $\lim\limits_{x \to x_0^+} f(x) = f(x_0)$ $\left(\lim\limits_{x \to x_0^-} f(x) = f(x_0)\right)$.

Ekvivalentní definice:

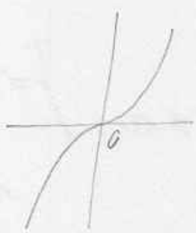
f je spojitá v bodě $x_0 \iff \forall \varepsilon > 0 \ \exists \delta > 0:$
$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

Funkce f je spojitá, když je spojitá v každém bodě.

Př: $f(x) = x^3$

Z definice $(\varepsilon - \delta)$ dokažte, že je f spojitá v bodě 0.

$x_0 = 0$, $f(0) = 0$



$\varepsilon = \frac{1}{2}$ $\quad \delta = ?$

$|x - 0| < \delta \Rightarrow |x^3| < \frac{1}{2}$

$|x^3| < \frac{1}{2} \Rightarrow |x| < \sqrt[3]{\frac{1}{2}}$

Položím $\delta = \sqrt[3]{\frac{1}{2}}$

$|x| < \sqrt[3]{\frac{1}{2}} \Rightarrow |x^3| < \frac{1}{2}$.

Libovolné $\varepsilon > 0$.

Položím $\delta = \sqrt[3]{\varepsilon} > 0$.

$|x| < \sqrt[3]{\varepsilon} \Rightarrow |x^3| < \varepsilon$ $\qquad \square$

Př: $f(x) = x - 1$ $\quad$ Dokažte spojitost v 2.

$x_0 = 2$

$f(2) = 1$

$\varepsilon > 0$ $\quad \delta = ?$

$|x - 2| < \delta \Rightarrow |f(x) - 1| < \varepsilon$

$|x - 1 - 1| = |x - 2| < \varepsilon$

Položím $\delta = \varepsilon$: $|x - 2| < \delta \Rightarrow |f(x) - 1| = |x - 2| < \delta = \varepsilon$. $\qquad \square$

Počítání s nekonečnem:

$$a \cdot \infty = \begin{cases} \infty & a > 0 \\ -\infty & a < 0 \end{cases} \qquad \frac{a}{\infty} = 0 \quad \forall a \in \mathbb{R}$$

$$a \cdot (-\infty) = \begin{cases} \infty & a < 0 \\ -\infty & a > 0 \end{cases} \qquad \frac{a}{-\infty} = 0 \quad \forall a \in \mathbb{R}$$

$$a + \infty = \infty + a = \infty + \infty = \infty \qquad \forall a \in \mathbb{R}$$

$$a - \infty = -\infty + a = -\infty - \infty = -\infty$$

$$\infty \cdot \infty = \infty \qquad (-\infty)\cdot(-\infty) = \infty \qquad (-\infty)\cdot \infty = -\infty$$

Nevíme co je: $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\infty - \infty$

- $\lim\limits_{x \to \infty} \frac{1}{x} = 0 \qquad \lim\limits_{x \to -\infty} \frac{1}{x} = 0$
- $\lim\limits_{x \to \infty} -\frac{1}{x} = 0 \qquad \lim\limits_{x \to -\infty} -\frac{1}{x} = 0$

- Určete limitu v nevlastních bodech funkce $f(x) = x^3 - 5x + 7$.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^3 - 5x + 7 = \lim_{x \to \infty} x^3\left(1 - \frac{5}{x^2} + \frac{7}{x^3}\right) = \infty(1 - 0 + 0) = \underline{\underline{\infty}}$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^3 - 5x + 7 = \lim_{x \to -\infty} x^3\left(1 - \frac{5}{x^2} + \frac{7}{x^3}\right) = -\infty$$

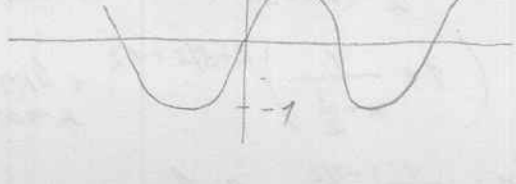
- Určete limitu v nevlastních bodech funkce $f(x) = \frac{2x^2 - 5x + 1}{-3x + 2}$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{2x^2 - 5x + 1}{-3x + 2} = \lim_{x \to \infty} \frac{x^2\left(2 - \frac{5}{x} + \frac{1}{x^2}\right)}{x\left(-3 + \frac{2}{x}\right)} =$$

$$= \lim_{x \to \infty} \frac{x\left(2 - \frac{5}{x} + \frac{1}{x^2}\right)}{-3 + \frac{2}{x}} = \infty \cdot \left(\frac{2 - 0 + 0}{-3 + 0}\right) = \underline{\underline{-\infty}}$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{2x^2 - 5x + 1}{-3x + 2} = \lim_{x \to -\infty} \frac{x\left(2 - \frac{5}{x} + \frac{1}{x^2}\right)}{-3 + \frac{2}{x}} = -\infty \cdot \left(\frac{2 - 0 + 0}{-3 + 0}\right) = \underline{\underline{\infty}}$$

- $\lim\limits_{x \to \infty} \sin x$ neexistuje



Spojitost se dá vyšetřovat i jiným způsobem a to pomocí limity.

f je spojitá v $x_0 \Leftrightarrow \lim\limits_{x \to x_0} f(x) = f(x_0)$

Pokud $\lim\limits_{x \to x_0-} f(x) = \lim\limits_{x \to x_0+} f(x) \Rightarrow \lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0^-} f(x) = \lim\limits_{x \to x_0^+} f(x)$

jestliže neplatí $\lim\limits_{x \to x_0} f(x) = f(x_0)$, potom buď:

a) $\lim\limits_{x \to x_0} f(x)$ neexistuje (vlastní)

- $\lim\limits_{x \to x_0^+} f(x) \neq \lim\limits_{x \to x_0^-} f(x) \Rightarrow x_0$ je bod nespojitosti 1. druhu
- alespoň 1 z jednostranných limit v bodě x_0 neexistuje (vlastní) $\Rightarrow x_0$ je bod nespojitosti 2. druhu

nebo

b) existuje (vlastní) limita $\lim\limits_{x \to x_0} f(x) = a$, avšak $a \neq f(x_0)$

- $f(x_0)$ není definováno
- $f(x_0)$ je definováno, avšak $f(x_0) \neq a$.

$\Rightarrow x_0$ je bodem odstranitelné nespojitosti funkce f

Př: $f(x) = x^3$ v $x_0 = 0$?

$\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x^3 = 0 = f(0)$

$\Rightarrow$ spojitost

$\lim\limits_{x \to a} f(x) = f(a)$

Př: $f(x) = x - 1$ v $x_0 = 2$?

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} (x-1) = 1 = f(2)$

$\Rightarrow$ spojitost v 2

Př: $f(x) = \begin{cases} \frac{1}{1+x} & x \geq 0 \\ 1-x^2 & x < 0 \end{cases}$ v $a = 0$?

V tomto případě budeme těžko počítat $\lim\limits_{x \to a}$ ale můžeme:

$$\left.\begin{array}{l} \lim\limits_{x \to 0_+} \frac{1}{1+x} = 1 \\ \lim\limits_{x \to 0_-} (1-x^2) = 1 \end{array}\right\} \lim_{x \to 0_+} f(x) = \lim_{x \to 0_-} f(x)$$

$$\Rightarrow \lim_{x \to 0} f(x) = \lim_{x \to 0_+} f(x) = \lim_{x \to 0_-} f(x) = \underline{\underline{1}}$$

$$f(0) = \frac{1}{0+1} = \underline{\underline{1}}$$

$\Rightarrow$ spojitost v 0.

Př: $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} \frac{1}{x} = +\infty$$

$$\lim_{x \to 0_-} f(x) = \lim_{x \to 0_-} \frac{1}{x} = -\infty$$

$$\nexists \lim_{x \to 0} f(x)$$

$\Rightarrow$ nespojitost v 0 - bod nespojitosti 2. druhu

Př: $f(x) = \begin{cases} x^2 & x \neq 0 \\ 2 & x = 0 \end{cases}$ v 0?

$$\left.\begin{array}{l} \lim\limits_{x \to 0_+} f(x) = \lim\limits_{x \to 0_+} x^2 = 0 \\ \lim\limits_{x \to 0_-} f(x) = \lim\limits_{x \to 0^-} x^2 = 0 \end{array}\right\} \lim_{x \to 0} x^2 = 0 \neq f(0) = 2$$

$\Rightarrow$ nespojitost v 0

- bod 0 je bod odstranitelné nespojitosti funkce f

Př: $f(x) = \begin{cases} x^4 & x \le 0 \\ 0 & x > 0 \end{cases}$ spojitost v 0?

$$\left.\begin{array}{l} \lim\limits_{x \to 0_+} f(x) = 0 \\ \lim\limits_{x \to 0_-} f(x) = \lim\limits_{x \to 0_-} x^4 = 0 \end{array}\right\} \Rightarrow \lim_{x \to 0} f(x) = 0 = f(0)$$

$\Rightarrow$ spojitost v 0.

Př: $f(x) = \begin{cases} x^2 & x \ge 1 \\ x & x < 1 \end{cases}$ spojitost v 1?

$$\left.\begin{array}{l} \lim\limits_{x \to 1_-} f(x) = \lim\limits_{x \to 1_-} x = 1 \\ \lim\limits_{x \to 1_+} f(x) = \lim\limits_{x \to 1_+} x^2 = 1 \end{array}\right\} \Rightarrow \lim_{x \to 1} f(x) = 1 = f(1)$$

$\Rightarrow$ spojitost v 1

Př: $f(x) = \begin{cases} x^2 & x \le 1 \\ 2x - 1 & x > 1 \end{cases}$ spojitost v 1?

$$\left.\begin{array}{l} \lim\limits_{x \to 1_-} f(x) = \lim\limits_{x \to 1_-} x^2 = 1 \\ \lim\limits_{x \to 1_+} f(x) = \lim\limits_{x \to 1_+} (2x-1) = 1 \end{array}\right\} \Rightarrow \lim_{x \to 1} f(x) = 1 = f(1)$$

$\Rightarrow$ spojitost

Př: $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

$\lim\limits_{x \to 0_-} f(x) = -1$

$\lim\limits_{x \to 0_+} f(x) = 1$

$\lim\limits_{x \to 0} f(x)$ neexistuje

$\Rightarrow$ nespojitost v 0

- nespojist 1. druhu.

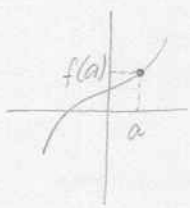
f(x) spojitá v bodě a?

$\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = A$

$\lim_{x\to a} f(x) = A$

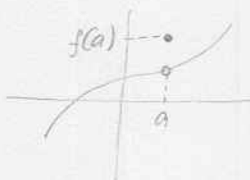
$f(a) = A$

f je spojitá v a



$f(a) \neq A$

a je bod odstranitelné nespojitosti

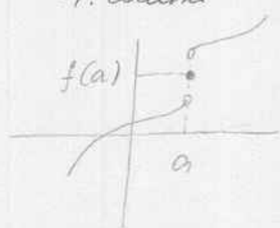


$\lim_{x\to a+} f(x) \neq \lim_{x\to a-} f(x)$

$\nexists \lim_{x\to a} f(x)$

$\lim_{\substack{x\to a+\\ x\to a-}} f(x) \neq \pm\infty$

nespojitost 1. druhu



$\lim_{x\to a+} f(x) = \infty$

$\lim_{x\to a-} f(x) = \infty$

nespojitost 2. druhu