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# Distributional chaos for triangular maps

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**Abstract**

In this paper we exhibit a triangular map $F$ of the square with the following properties: (i) $F$ is of type $2^\infty$ but has positive topological entropy; we recall that similar example was given by Kolyada in 1992, but our argument is much simpler. (ii) $F$ is distributionally chaotic in the wider sense, but not distributionally chaotic in the sense introduced by Schweizer and Smítal [Trans. Amer. Math. Soc. 344 (1994) 737]. In other words, there are lower and upper distribution functions $\Phi_{xy}$ and $\Phi^*_{xy}$ generated by $F$ such that $\Phi^*_{xy} \equiv 1$ and $\Phi_{xy}(0_+) < 1$, and no distribution functions $\Phi_{uv}$, and $\Phi^*_{uv}$ such that $\Phi^*_{uv} \equiv 1$ and $\Phi_{uv}(t) = 0$ whenever $0 < t < \epsilon$, for some $\epsilon > 0$. We also show that the two notions of distributional chaos used in the paper, for continuous maps of a compact metric space, are invariants of topological conjugacy.


## 1. Introduction and preliminaries

Let $I = [0,1]$ be the unit interval. By a *triangular map* we mean a continuous map $F : I^2 \to I^2$ of the form $F(x,y) = (f(x), g_x(y))$. The map $f$ is called the *base* for $F$, $g_x$ is a map from the *layer* $I_x = \{x\} \times I$ to $I$. Denote by $\rho$ the metric on the unit square $I^2$ given by $\rho((x_1,y_1),(x_2,y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.

Let $f$ be a map from a compact metric space $(M,d)$ into itself. We say that $f$ is of *type* $2^\infty$ if the periods of the periodic orbits of $f$ are the numbers $2^n$, for any integer $n \leqslant 0$. For any pair $(x,y)$ of points in $M$ and any positive integer $n$, define a distribution function $\Phi^{(n)}_{xy} : \mathbf{R} \to [0,1]$ by

$$\Phi^{(n)}_{xy}(t) = \frac{1}{n}\#\{0 \leqslant i \leqslant n-1;\ d(f^i(x), f^i(y)) < t\}.$$

Obviously, $\Phi^{(n)}_{xy}(t)$ is a non-decreasing function, $\Phi^{(n)}_{xy}(t) = 0$ for $t \leqslant 0$ and $\Phi^{(n)}_{xy}(t) = 1$ for $t$ greater than the diameter of $M$. Put

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi^{(n)}_{xy}(t) \quad \text{and} \quad \Phi^*_{xy}(t) = \limsup_{n\to\infty} \Phi^{(n)}_{xy}(t).$$

The function $\Phi_{xy}$ is called the *lower distribution*, and $\Phi^*_{xy}$ the *upper distribution* of $x$ and $y$. Obviously, $\Phi_{xy}(t) \leqslant \Phi^*_{xy}(t)$ for any real $t$. If $\Phi_{xy}(t) < \Phi^*_{xy}(t)$ for all $t$ in an interval, we simply write $\Phi_{xy} < \Phi^*_{xy}$.

If there is a pair of points $(x,y)$ in $M$ such that

$$\Phi^*_{xy} \equiv 1 \quad \text{and} \quad \Phi_{xy}(t) = 0 \quad \text{for some } t > 0 \tag{1}$$

or

$$\Phi^*_{xy} \equiv 1 \quad \text{and} \quad \Phi_{xy} < \Phi^*_{xy}, \tag{2}$$

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then we say that $f$ is *distributionally chaotic in the strict*, *resp. wider sense* (briefly. *DC1*, *resp. DC2*). Recall that *DC1* was originally introduced in [5], *DC2* is a generalization.

It is known that, for a continuous map of the unit interval, the notions of *DC1, DC2* and positive topological entropy (simply, *PTE*) are equivalent [5]. We give here an example of a triangular map with *PTE* which is *DC2*, but not *DC1*. Since there are *DC1* triangular maps with zero topological entropy [2] we have proved that, in the class of triangular maps, *PTE* and *DC1* are independent properties.

## 2. Construction of the map

**Theorem 1.** *There exists a triangular map $F$ of type $2^\infty$ with positive topological entropy such that $F$ is DC2 but is not DC1.*

**Proof.** STAGE 1. Construction of a triangular map $F$ of type $2^\infty$. It is inspired by the map from [3], but our argument is more simple.

Let $f(x) = \lambda x(1-x)$, where $\lambda = 3,569\ldots$ is such that $f$ is of type $2^\infty$. It is well-known that $f$ has exactly one infinite $\omega$-limit set $Q$ which is a perfect set (i.e., non-empty, compact, without isolated points), two fixed points and, for any $n > 0$, a unique periodic orbit of period $2^n$. Then there is a sequence $\{I_n\}_{n=0}^\infty$ of minimal compact periodic intervals such that

(i) $I_n$ has period $2^n$, for any $n = 0, 1, \ldots$,
(ii) $\bigcap_{n=0}^\infty \mathrm{Orb}(I_n) = Q$,
(iii) $\bigcap_{n=0}^\infty I_n = \{c\}$, for some $c \in Q$.

Denote by $p_n$ the unique periodic point of period $2^n$ belonging to $I_n$. (The existence of the intervals $I_n$ with the above mentioned properties is well-known, even for a general continuous map of the interval of type $2^\infty$, cf., e.g., [6].)

Now, we can proceed to the construction of the layer functions. Let

$$g_0 \equiv 0, \quad g_c \equiv 0 \quad \text{and} \quad g_{pn} \equiv 0 \quad \text{for any } n = 0, 1, \ldots \tag{3}$$

Let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of positive integers (it will be specified later). Let $\tau$ be the tent map, $\tau(x) = 1 - |1 - 2x|$. Put

$$g_x = \frac{1}{2^k}\tau \quad \text{for } x \in Q \cap (I_{n_k} \setminus I_{n_{k+1}}), \tag{4}$$

and let

$$g_x(0) = 0 \quad \text{for any } x \in I. \tag{5}$$

Finally, extend $F$ continuously to the whole of $I^2$. Then $F$ is of type $2^\infty$. Indeed, the projection of a periodic point of a triangular map to the $x$-axis is a periodic point of the base map $f$. Thus, by (3) and (5),

$$Per(F) = Per(f) \times \{0\}, \tag{6}$$

hence $f$ and $F$ have the same periods of periodic points, i.e., $F$ is of type $2^\infty$.

STAGE 2. We show that a particular choice of the sequence $\{n_k\}$ implies positive topological entropy of $F$. Indeed, we have $h(F) = h(F|_{C(F)})$, where $C(F) \subset (Q \times I) \cup (Per(f) \times I)$ is the centre of $F$. By (3) and (5), every trajectory in $Per(f) \times I$ is eventually periodic, hence $h(F|_{Per(f)\times I}) = h(F|_{Per(F)}) = 0$ and consequently, $h(F) = h(F|_{Q\times I})$. By Bowen formula [1],

$$h(f) + \sup_{x\in Q} h(F, I_x) \geqslant h(F) \geqslant \max\{h(f), \sup_{x\in Q} h(F, I_x)\}, \tag{7}$$

where $h(F, I_x)$ is topological entropy computed from trajectories starting from the fibre $I_x$. In our case, however, $h(f) = 0$ hence $h(F) = \sup_{x\in Q} h(F, I_x)$. To compute $h(F, I_x)$ we apply the variational principle. For an $x \in Q$ and any positive integer $n$ denote by $V(n, x)$ the number of laps of the map $F^n$ restricted to the fibre $I_{f^n(x)}$. Then (see [4])

$$h(F) = h(F|_{Q\times I}) = \sup_{x\in Q} \lim_{n\to\infty} \frac{1}{n} \log V(n, x). \tag{8}$$

Since, for $x \in Q$, the fibre maps are the tent maps or their multiples by $2^{-k}$, $k > 0$, we have obviously $V(n, x) \leqslant 2^n$. To get the lower estimation note that when the trajectory of $x$ at some time visits $I_{n_k}$ then the next $2^{n_k}$ applications of the tent map are necessary to recover the diminishing of the layer map caused by the factor $1/2^k$, cf. (4). Denote by $v(x,i) = \|F^i|_{I_{f^i(x)}}\|$ the norm of the layer map of $F^i$ over $f^i(x)$. We are interested in the size of the set $S_n(x) = \{0 \leqslant i < n; v(x,i) < 1\}$ since $V(n,x) = 2^{n-\#S_n(x)}$. Put $\eta(x) = \liminf_{n\to\infty} \frac{1}{n}\#S_n(x)$, $\sigma(x) = \limsup_{n\to\infty} \frac{1}{n}\#S_n(x)$. Then, by (8),

$$h(F|_{Q\times I}) \geqslant (\log 2)\sup_{x\in Q}(1-\eta(x)) \geqslant (\log 2)\sup_{x\in Q}(1-\sigma(x)). \tag{9}$$

Thus to get $h(F) > 0$ it suffices to make $\sigma(x) < 1$, for some $x \in Q$. It is easy to compute $\sigma(x)$: the relative density of times when the trajectory of $x$ visits $I_{n_1} \setminus I_{n_2}$ is $2^{-n_1} - 2^{-n_2}$, and every such visit is recovered by the next application of the tent map. If the trajectory visits $I_{n_2} \setminus I_{n_3}$ then two applications of the tent map are necessary to recover the diminishing., etc. Thus, we have

$$\sigma(x) = \left(\frac{1}{2^{n_1}} - \frac{1}{2^{n_2}}\right) + 2\left(\frac{1}{2^{n_2}} - \frac{1}{2^{n_3}}\right) + \cdots + k\left(\frac{1}{2^{n_k}} - \frac{1}{2^{n_{k+1}}}\right) + \cdots,$$

hence

$$\sigma(x) = \frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \cdots + \frac{1}{2^{n_k}} + \cdots \tag{10}$$

So, if the sequence $\{n_k\}_{k=1}^{\infty}$ is such that $\sigma(x)$ given by (10) is less than 1 then $F$ has positive topological entropy.

STAGE 3. Now we can show that $F$ is not $DC1$. For any integer $k > 0$,

$$v(x,i) < \frac{1}{2^k} \quad \text{whenever } f^i(x) \in I_{n_{k+1}}. \tag{11}$$

Let $S_{k,n}(x) = \{0 \leqslant i < n; v(x,i) < 1/2^k\}$, and $\eta_k(x) = \liminf_{n\to\infty} \frac{1}{n}\#S_{k,n}(x)$. Since $I_{n_{k+1}}$ is periodic, (11) implies

$$\eta_k(x) \geqslant \frac{1}{2^{n_{k+1}}} > 0. \tag{12}$$

Now assume contrary to what we wish to show that there are distinct points $u = (x,y)$, $v = (x',y')$ in $I^2$ such that $\Phi^*_{u,v} \equiv 1$, and $\Phi_{u,v}(\epsilon) < 1$ for some $\epsilon \in (0,1)$. We may assume without loss of generality that $F$ has the following additional properties (cf. construction in Stage 1):

(i) $g_t \equiv 0$ for any $t$ in a neighborhood of $p_n$, for any $n$.
(ii) $c$ is an endpoint of an interval $J$ complementary to $Q$.
(iii) $c \neq \min Q$, $c \neq \max Q$.
(iv) There is a neighborhood $U$ of $c$ such that $g_t \equiv 0$ for $t \in U \cap J$.

Then

$$x \notin Q \text{ implies } F^n(u) \in ((I\setminus Q)\times\{0\}) \cup (Q\times I) \quad \text{for some } n > 0. \tag{13}$$

This follows by (i) if $\omega_f(x)$ is a periodic orbit, and by (ii)–(iv) otherwise. Similar argument is true for $x'$.

By $\Phi^*_{uv} \equiv 1$, $\omega_f(x) = \omega_f(x)' =: \tilde{\omega}$. If $\tilde{\omega}$ is a periodic orbit then, by (13), the trajectory of either $u$ or $v$ is eventually in $(I\setminus Q)\times\{0\}$, and since $F$ restricted to $(I\setminus Q)\times\{0\}$ is isomorphic to $f$ restricted to $I\setminus Q$, we have, $\Phi^*_{uv} = \Phi_{uv}$. If $\tilde{\omega} = Q$ and $x \notin Q$ then, again by (13) we may replace $u$ by a point in $Q\times I$ without changing $\Phi^*_{uv}$ and $\Phi_{uv}$. Hence it suffices to consider the case $u, v \in Q\times I$ and consequently, $x = x' \in Q$ (since in the case $x \neq x'$ we immediately obtain $\Phi^*_{uv} \not\equiv 1$).

Since $\xi(u,v,1/2^k,n) = \#\{0 \leqslant i < n; \rho(F^i(u),F^i(v)) < 1/2^k\} \geqslant \#\{0 \leqslant i < n; v(x,i) < 1/2^k\} = S_{k,n}$, (12) implies $\Phi_{uv}(1/2^k) \geqslant \eta_k(x) > 0$, for any $k \geqslant 0$. Consequently, $F$ is not $DC1$.

Finally, let $\tilde{y} \in I$ be such that the points 0 and $\tilde{y}$ generate $DC1$ for the tent map. Then putting $u = (x,0)$, $v = (x,\tilde{y})$ we get $\Phi^*_{uv} \equiv 1 > \Phi_{uv}$ and hence $F$ is $DC2$, for a suitable choice of $x$. □

## 3. DC1 and DC2 are invariants of topological conjugacy

Our main result in this section is the following theorem.

**Theorem 2.** *Let $\varphi$, $\psi$ be topologically conjugate continuous maps of a compact metric space $(X,\rho)$. Then $\varphi$ is $DC1$ or $DC2$ if and only if $\psi$ is $DC1$ or $DC2$, respectively.*

**Proof.** Let $h$ be a homeomorhism of $x$ conjugating $\varphi$ and $\psi$ such that $\psi = h \circ \varphi \circ h^{-1}$. By the continuity of $h$, for any $\epsilon > 0$ there is an arbitrarily small $\delta > 0$ such that, for any $u, v \in X$.

$$\rho(u, v) < \delta \text{ implies } \rho(h(u), h(v)) < \epsilon.$$

Then

$$\rho(\varphi^n(u), \varphi^n(v)) < \delta \text{ implies } \rho(h \circ \varphi^n(u), h \circ \varphi^n(v)) < \epsilon$$

and since $h \circ \varphi^n = \psi^n \circ h$,

$$\rho(\varphi^n(u), \varphi^n(v)) < \delta \text{ implies } \rho(\psi^n \circ h(u), \psi^n \circ h(v)) < \epsilon.$$

Consequently

$$\Phi^*_{uv}(\delta) \leqslant \Psi^*_{h(u)h(v)}(\epsilon), \tag{14}$$

where $\Phi^*_{uv}$ or $\Psi^*_{uv}$ are upper distribution functions of $\varphi$ or $\psi$ respectively. Similarly, by the continuity of $h^{-1}$, for any $\epsilon > 0$ there is an arbitrarily small $\delta > 0$ such that, for any $u, v \in X$,

$$\rho(h(u), h(v)) < \delta \text{ implies } \rho(u, v) < \epsilon$$

and consequently,

$$\Psi_{h(u)h(v)}(\delta) \leqslant \Phi_{uv}(\epsilon). \tag{15}$$

Now if $\Phi^*_{uv} \equiv 1$ then, by (14), $\Psi_{h(u)h(v)} \equiv 1$ If, on the other hand, $\Phi_{uv}(\epsilon) = 0$ then, by (15), $\Psi_{h(u)h(v)}(\delta) = 0$ and consequently, if $\varphi$ is $DC1$ then $\psi$ is also $DC1$. Moreover, if $\Phi_{uv}(\epsilon) < 1$ then, again by (15), $\Psi_{h(u)h(v)}(\delta) < 1$ and hence, if $\varphi$ is $DC2$ then also $\psi$ is $DC2$. $\square$

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