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The three versions of distributional chaos ☆

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Abstract

The notion of distributional chaos was introduced by Schweizer and Smítal [Trans. Amer. Math. Soc. 344 (1994) 737] for continuous maps of the interval. However, it turns out that, for continuous maps of a compact metric space three mutually nonequivalent versions of distributional chaos, *DC1–DC3*, can be considered. In this paper we consider the weakest one, *DC3*. We show that *DC3* does not imply chaos in the sense of Li and Yorke. We also show that *DC3* is not invariant with respect to topological conjugacy. In other words, there are lower and upper distribution functions Φ_{xy} and Φ^*_{xy} generated by a continuous map f of a compact metric space (M, ρ) such that $\Phi^*_{xy}(t) > \Phi_{xy}(t)$ for all t in an interval. However, f on the same space M, but with a metric ρ' generating the same topology as ρ is no more *DC3*.

Recall that, contrary to this, either *DC1* or *DC2* is topological conjugacy invariant and implies Li and Yorke chaos (cf. [Chaos, Solitons & Fractals 21 (2004) 1125]).

1. Introduction and preliminaries

Let f be a map from a compact metric space (M, ρ) into itself. For any pair (x, y) of points in M and any positive integer n, define a distribution function $\Phi^{(n)}_{xy} : \mathbf{R} \to [0, 1]$ by

$$\Phi^{(n)}_{xy}(t) = \frac{1}{n}\#\{0 \leqslant i \leqslant n-1; \rho(f^i(x), f^i(y)) < t\}.$$

Then $\Phi^{(n)}_{xy}(t)$ is a non-decreasing function, $\Phi^{(n)}_{xy}(t) = 0$ for $t \leqslant 0$, and $\Phi^{(n)}_{xy}(t) = 1$ for t greater than the diameter of M. Put

$$\Phi_{xy}(t) = \liminf_{n\to\infty} \Phi^{(n)}_{xy}, \quad \text{and} \quad \Phi^*_{xy}(t) = \limsup_{n\to\infty} \Phi^{(n)}_{xy}(t).$$

Then Φ_{xy} is called the *lower distribution function*, and Φ^*_{xy} the *upper distribution function* of x and y. Obviously, $\Phi_{xy}(t) \leqslant \Phi^*_{xy}(t)$ for any real t. If $\Phi_{xy}(t) < \Phi^*_{xy}(t)$ for all t in an interval, we simply write $\Phi_{xy} < \Phi^*_{xy}$.

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If there is a pair of points (x, y) in M such that

$$\Phi^*_{xy} \equiv 1 \quad \text{and} \quad \Phi_{xy}(t) = 0 \quad \text{for some } t > 0, \tag{1}$$

or

$$\Phi^*_{xy} \equiv 1 \quad \text{and} \quad \Phi_{xy} < \Phi^*_{xy}, \tag{2}$$

or

$$\Phi^*_{xy} > \Phi_{xy}, \tag{3}$$

then we say that f exhibits *distributional chaos of type 1–3, briefly*, DC1, DC2, or DC3, respectively. Recall that *DC1* was originally introduced in [5], *DC2 and DC3* are generalizations.

It is known that, for a continuous map of the interval, the notions of *DC1–DC3* and positive topological entropy are equivalent [5]. It also follows by the definition that, for a continuous map of a compact metric space, *DC1* implies *DC2,* and *DC2* implies *DC3*. In this paper we prove two results: First we exhibit a continuous map of a compact metric space M which is *DC3* and distal. Hence it cannot be chaotic in the sense of Li and Yorke and consequently, it fails to be *DC2*. Moreover, we show that there is an another metric on M generating the same topology such that f is no more *DC3* with this new metric. Thus, *DC3* is not invariant with respect to the topological conjugacy. Contrary to this, *DC1* and *DC2* are invariants [6]. For related results we refer also to [2,4], or [1].

For an interesting application of *DC3* see [3]. There the measure of chaos of continuous maps of a compact space X generated by a pair $(x, y) \in X \times X$ is defined as $\sup_\rho \int_0^1 (\Phi^*_{xy}(t) - \Phi_{xy}(t))\,dt$, the supremum being taken over all metrics ρ with $\rho(X) = 1$ generating the topology of X.

2. Proof of the main results

Theorem 1. *There is a continuous map F of a compact metric space M which is DC3 but not chaotic in the sense of Li and Yorke.*

Proof. Let X be the set $\{0,1\}^\infty$ of sequences of two symbols, equipped with the metric ρ_X of pointwise convergence. Let Y be the subset of the complex plane consisting of the unit circle, and a singleton $\{2\}$; i.e., $z \in Y$ if and only if either $|z| = 1$ or $z = 2$. Finally, let $M = X \times Y$, and let ρ be the max metric on M given by $\rho((x_1, y_1), (x_2, y_2)) = \max\{\rho_X(x_1, x_2), |y_1 - y_2|\}$. For any $(x, y) \in M$ we let $F(x, y) = (f(x), g_x(y))$, where the maps f, g_x are given below. Thus, F is a triangular map. The map f is the adding machine on Y. This means that $f(x)$ is obtained from x by adding $1 = 1000\ldots$, the adding is taken mod 2 from the left to the right. (for more information concerning adding machine cf., e.g., [2].)

The map g_x is given by

$$g_x(y) = e^{2\pi i(p(x)+y)} \quad \text{if } |y| = 1 \text{ and } g_x(1) = 1, \tag{4}$$

where the map $p: X \to R$ is given later. Thus, for any $x \in X$, g_x considered as a map $Y \to Y$ rotates the circle, and fixes the point 1. Hence, neither f nor g_x has a pair of distinct proximal points. In other words, the system generated by F is distal and consequently, F is not chaotic in the sense of Li and Yorke. To complete the argument it suffices to define a continuous map $p: X \to R$ such that F is $DC3$.

Let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of positive integers (it will be specified later). For any positive integer k let φ_k be a continuous map $\{0,1\}^{n_k} \to [0,1]$, given by

$$\varphi_k(x) = 0 \quad \text{if } 2^{k-1} \leqslant |x| < 2^{n_k} - 2^k \quad \text{and} \quad \varphi_k(x) = \frac{1}{2^k} \text{ otherwise}, \tag{5}$$

where $|x|$ is the *evaluation* of $x = x_1x_2\cdots x_{n_k} \in \{0,1\}^{n_k}$ given by

$$|x_1x_2\cdots x_{n_k}| = x_1 + 2x_2 + 2^2x_3 + \cdots + 2^{n_k-1}x_{n_k}. \tag{6}$$

Any $x \in X$ can be uniquely written in the block form

$$x = x^1x^2x^3\cdots x^k\cdots, \tag{7}$$

where x^k is the block of n_k symbols in x. Now we let $p(1^\infty) = 0$ and, for $x \neq 1^\infty$, let $p(x) = \varphi_k(x)$ if x^k is the first block in (7) containing at least one zero digit. Thus,

$$\text{for } x \neq 1^\infty,\ p(x) = \varphi_k(x) \text{ whenever } x^k \neq 1^{n_k} \text{ and } x^i = 1^{n_i} \quad \text{for } i < k. \tag{8}$$

By (5), the norm $|\varphi_k|$ of φ_k is 2^{-k}, hence (8) implies the continuity of p. It remains to show that F is $DC3$. We show that the sequence $\{n_k\}$ can be chosen such that

$$\Phi^*_{uv}(t) > \Phi_{uv}(t) \quad \text{if } u = (0^\infty, -1),\ v = (0^\infty, 2),\ \text{and } 2 < t \leqslant \frac{5}{2}. \tag{9}$$

For a nonnegative integer i, let $y_i \in Y$ be the second coordinate of $F^i(u)$. For any positive integer k, put $m_k = n_1 + n_2 + \cdots + n_k$, and

$$L_k = \#\{0 \leqslant i < 2^{m_k}; y_i = 0\}, \quad U_k = \#\{0 \leqslant i < 2^{m_k}; y_i = 1/2\}. \tag{10}$$

By (5) and (8) we have $y_0 y_1 \cdots y_{2^{n_1}-1} = 0\frac{1}{2}\frac{1}{2}\cdots\frac{1}{2}0 =: B_1$. Thus,

$$L_1 = 2, \quad U_1 = 2^{n_1} - 2. \tag{11}$$

To estimate L_2 note that, by (5) and (8), the block of y_i, $0 < i < 2^{m_2}$, can be written in the block form $B_1 B_2 \cdots B_{2^{n_2}}$, each B_i having 2^{n_1} digits. The block B_1 is described above. B_2 is obtained from B_1 by adding $\frac{1}{4}$ to each digit, and B_3 from B_2 again by adding $\frac{1}{4}$ to each digit. All other blocks B_i, but the last two, equal to B_3, by (5). Since the values y_i are taken mod 1, we can see that any of the blocks B_i, $2 < i < 2^{n_2} - 2$ is "inverse" to B_1 such that each zero is replaced by $\frac{1}{2}$, and each $\frac{1}{2}$ by zero. Hence,

$$L_2 \geqslant (2^{n_1} - 2)(2^{n_2} - 4). \tag{12}$$

Similarly we obtain

$$U_3 \geqslant (2^{n_1} - 2)(2^{n_2} - 4)(2^{n_3} - 8), \tag{13}$$

etc. Denote by P_k the product $(2^{n_1} - 2)(2^{n_2} - 4)(2^{n_3} - 8) \cdots (2^{n_k} - 2^k)$. By induction,

$$U_{2k-1} \geqslant P_{2k-1}, \quad L_{2k} \geqslant P_{2k}, \qquad k = 1, 2, 3, \ldots \tag{14}$$

It is well known that for any $\alpha \in (0,1)$ there is an increasing sequence $\{n_k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} P_k = \alpha. \tag{15}$$

Then, for any $t \in (2, \frac{5}{2}]$, by (5), (9), (10), (14) and (15),

$$\Phi^*_{uv}(t) \geqslant \alpha, \quad \Phi_{uv}(t) \leqslant 1 - \alpha, \tag{16}$$

which implies $DC3$, for $\alpha > \frac{1}{2}$. □

Theorem 2. *The notion of DC3 is not invariant with respect to topological conjugacy for continuous maps on compact metric spaces.*

Proof. Let M be the same set as in the proof of Theorem 1, but with metric ρ' such that $\rho'((x_1, y_1), (x_2, y_2)) = \max\{\rho_X(x_1, x_2), \nu(y_1, y_2)\}$. Here ν is a metric on Y such that $\nu(y_1, y_2) = |y_1 - y_2|$ if $|y_1| = |y_2| = 1$, and $\nu(y, 2) = 1$ for any $y \neq 2$. Now if h is the identity map from (M, ρ) onto (M, ρ'), then h is a homeomorphism and, if F is the map from Theorem 1 then the conjugate map $F' = h \circ F \circ h^{-1}$ is not $DC3$. □

References

[1] Balibrea F, Reich L, Smítal J. Iteration theory: dynamical systems and functional equations. Int J Bifur Chaos 2003;13:1627–48.

[2] Forti G-L, Paganoni L, Smítal J. Dynamics of homeomorphisms on minimal sets generated by triangular mappings. Bull Austral Math Soc 1999;59:1–20.

[3] Bobok J. Chaos in countable dynamics. Topol Appl 2002;126:207–16.

[4] Schweizer B, Sklar A, Smítal J. Distributional (and other) chaos and its measurement. Real Anal Exchange 2000/2001;26:495–524.

[5] Schweizer B, Smítal J. Measures of chaos and a spectral decomposition of dynamical systems on the interval. Trans Amer Math Soc 1994;344:737–854.

[6] Smítal J, Štefánková M. Distributional chaos for triangular maps. Chaos, Solitons & Fractals 2004;21:1125–8.