Relativistic Coulomb problem for modified Stueckelberg equation

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In this paper we found explicit eigenvalues and eigenstates of discrete spectrum for the modified Stueckelberg equation (written in second order formalism) for the case of external Coulomb field. We considered the case of arbitrary gyromagnetic ratio.

1 Introduction

Obtaining of consistent theory of the spin-1 particles was a problem of interest for a long time. The Yang – Mills theory (with spontaneously broken symmetry) solved this problem for fundamental physics. But at the level of applied physics we also meet massive charged spin-1 particles (e.g. spin-1 states of helium nuclei) which require adequate (but phenomenological) description. That is one of the reasons for growing interest to this topic in last years (see discussion in [1]).

We would like to stress that the equations we consider are non-secondly quantized ones. Hence, the wave function of our particle must be understood as the matrix element of the quantized field operator between the one-particle state and the vacuum (as for the case of spinor field in [2]) at the approximation when both quantum character of external electromagnetic field and the effects of creation and annihilation of the particles may be neglected. This approximation seems to be natural since the quantum electrodynamics of the spin-1 particles is non-renormalizable [3] (and hence usual perturbation theory is not valid), that's why the evident first step in its study must be "semiclassical" (i.e. non-secondly quantized) theory similar to the one we consider here.

But even within the frame of such theory it seemed to be no good equation for spin-1 particles due to the following difficulties:

- introduction of minimal interaction into the free spin-1 Proca equations lead to the incompleteness of the set of regular solutions in Coulomb problem (as it was pointed out by Corben and Schwinger [4] and also by Tamm [5]) and to the absence of spin-orbital interaction in nonrelativistic limit [6];
- introduction of anomalous interaction leads to the Corben – Schwinger anomaly [4] for the spin-1 particle with gyromagnetic ratio $g = 2$ in Coulomb field (refer to section 5 below);
- Tsai et al. observed that any anomalous interaction for a spin-1 particle leads to complex energies in sufficiently strong constant homogeneous magnetic field (for more details see [7] and references therein);
- there was also the no-go theorem of Vijayalakshmi et al. [8] stating that either any anomalous interaction or its nonlinear (in electromagnetic field strength) generalisations can get over the above difficulties only at the expense of loosing the causality.

Beckers,Debergh and Nikitin [7] have overcome difficulties with no-go theorem and imaginary energy eigenvalues by offering a new kind of interaction with external electromagnetic field. Their results suggest the following question : does this new interaction eliminate Corben–Schwinger type anomalies in the Coulomb problem?

In this paper we partially answer it. In order to avoid rather complicated dealing with constraints usually arising in spin-one theory we consider modified à-la Beckers, Debergh and Nikitin [7] Stueckelberg equation in external Coulomb field. As we show below, this model hasn't the above-mentioned anomalies in Coulomb problem.

Let us consider the generalized Stueckelberg equation

$$(D_\mu D^\mu + m_{eff}^2)B^\nu + iegF_\rho^\mu B^\rho = 0, \tag{1}$$

where

$$m_{eff}^2 = M^2 + k_2 \mid e^2 F_{\mu\nu}F^{\mu\nu}/2 \mid^{1/2}, \quad D_\mu = \partial_\mu + ieA_\mu \tag{2}$$

Notice that this equation is similar to the two-particle equation for positronium from [1] up to the replacement of m_{eff}^2 by m^2 and choice of value of g.

We use $\hbar = c = 1$ unit system and following notations : small Greek letters denote indices running from 0 to 3; the metrics of Minkowski space is $g_{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$; we write the four-vector as $B^\mu = (B^0, \mathbf{B})$, where bold letter denotes its three-vector part; coordinates and derivatives are $x^\mu = (t, \mathbf{r})$, $\partial_\mu = \partial/\partial x^\mu$; A^μ are potentials of external electromagnetic field; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is electromagnetic field strength; e, g and M are respectively charge, gyromagnetic ratio and mass of the particle described by (1). Its wave function is given by the four-vector B^μ. In free case [7] this particle has two possible spin states : spin-0 and spin-1 ones with the same mass M.

Equation (1) is a minor generalization of modified Stueckelberg equation from [7] (authors of [7] consider only $g = 2$ case). Expression (2) for m_{eff}^2 differs from that of [7] by the presence of the module sign, which is necessary in order to avoid complex energy eigenvalues in Coulomb field, as it is straightforward to check (else μ_i (11) and, consequently, energy eigenvalues of discrete spectrum E^{inj} will be complex).

2 Coulomb field

The 4-potential, corresponding to the Coulomb field of attraction, is:

$$\mathbf{A} = 0, A^0 = -Ze/r, Z > 0. \tag{3}$$

Since it is static and spherically symmetric, energy E and total momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ ($\mathbf{L}$ is an angular momentum and $\mathbf{S}$ is a spin) are integrals of motion.

Let us decompose the wavefunctions of stationary states by the basis of common eigenfunctions of $\mathbf{J}^2$ and $\mathbf{J}_z$ with eigenvalues $j(j+1)$ and m respectively. The corresponding eigenmodes are:

$$B^0 = iF(r)Y_{jm}, \mathbf{B} = B^{(-1)}(r)\mathbf{Y}_{jm}^{(-1)} + B^{(0)}(r)\mathbf{Y}_{jm}^{(0)} + B^{(1)}(r)\mathbf{Y}_{jm}^{(1)} \tag{4}$$

where $\mathbf{Y}_{jm}^{(\lambda)}$ are spherical vectors (see [9] for their explicit form) and Y_{jm} are usual spherical functions. For $j = 0$ $B^{(0)} = B^{(1)} \equiv 0$ [9].

After substitution of (3) and (4) into (1) we obtain the following equations :

$$\begin{array}{ll} \mathrm{T}W = (2/r^2)\mathrm{Q}W, & j = 0 \\ \mathrm{T}V = (2/r^2)\mathrm{P}V, \quad \mathrm{T}B^{(0)} = 0, & j \neq 0, \end{array} \tag{5}$$

where

$$\beta = Ze^2, B^{(0)} \equiv K^{(0)},$$

$$\mathrm{T} = (E + \beta/r)^2 + d^2/dr^2 + (2/r)d/dr - j(j+1)/r^2 - m_{eff}^2 \tag{6}$$

$$V = (FB^{(-1)}B^{(1)})^{\dagger}, \quad W = (FB^{(-1)})^{\dagger}$$

$$\mathrm{P} = \begin{pmatrix} 0 & -b & 0 \\ b & 1 & -a \\ 0 & -a & 0 \end{pmatrix}, \quad \mathrm{Q} = \begin{pmatrix} 0 & -b \\ b & 1 \end{pmatrix}$$

($\dagger$ denotes matrix transposition, $a = \sqrt{j(j+1)}$, $b = \beta g/2$).

Matrices P and Q are diagonalizable; the eigenvalues of Q are

$$\lambda_{1,2} = 1/2 \pm \sqrt{1/4 - (\beta g/2)^2} \tag{7}$$

and those of P

$$\lambda_{1,2} = 1/2 \pm \sqrt{(j+1/2)^2 - (\beta g/2)^2}, \quad \lambda_3 = 0 \tag{8}$$

After diagonalization of P and Q equations (5) take the form (for convenience we denote $\lambda_0 \equiv \lambda_3$)

$$\mathrm{T}K^{(i)} = (2\lambda_i/r^2)K^{(i)} \qquad i = 0, 1, 2, 3 \tag{9}$$

Equations (9) formally coinside with ones for radial functions of Coulomb problem for arbitrary spin [1]. Using the results of [1], we easily obtain the corresponding energy eigenvalues of discrete spectrum:

$$E^{inj} = M/\sqrt{1 + \beta^2/(n + \mu_i + 1)^2} \tag{10}$$

where $n = 0, 1, 2, \ldots; j = 0, 1, 2 \ldots; i = 0, 1, 2, 3$ and

$$\mu_i = -1/2 + \sqrt{(j+1/2)^2 - \beta^2 + 2\lambda_i + k_2\beta} \tag{11}$$

(index i corresponds to the following eigenmode : $K^{(i)} \neq 0$, other functions $K^{(l)} = 0$; for $j = 0$ we have only two branches, corresponding to $i = 1, 2$). Branches of the spectrum for $i = 0$ and $i = 3$ are completely identical, i.e. we meet here twofold degeneracy.

The discrete spectrum eigenfunctions are

$$K^{(i)nj} = c^{inj} x^{\mu_i} exp(-x/2) L_n^{\mu_i}(x) \tag{12}$$

where c^{inj} are normalization constants, $x = 2r\sqrt{M^2 - E^2}$, L_n^{α} are Laguerre polynomials.

3 Extra degeneracy for $k_2 = 0, j > 0, g = 2$

Let us demonstrate that in this case levels (10) have fourfold degeneracy for $n > 1$. To prove it we notice that for $k_2 = 0, j > 0, g = 2$

$$\mu_1 = \lambda_1, \mu_2 = \lambda_1 - 2, \mu_0 = \lambda_1 - 1 \tag{13}$$

and substitute these μ_i in (10). We obtain four completely identical sets of levels (for $n > 1$).

When we exclude one of two identical branches with $i = 0$ and $i = 3$, we obtain threefold degeneracy, typical for the so-called parasupersymmetry [11]. We shall discuss this topic with more details in our following papers.

4 Elimination of spin-0 component

It is well known [2] that in free case ($e = 0$) we can exlude spin-0 component by setting the condition

$$\partial_\mu B^\mu = 0, \tag{14}$$

In the case of Coulomb field for $k_2 = 0, j > 0, g = 2$ it is straightforward to check that the condition

$$D_\mu B^\mu = 0 \tag{15}$$

is compatible with the equations of motion (9). Indeed,

$$D_\mu B^\mu = (EK^{(3)} + R^{(1)} + R^{(2)})Y_{jm}, \tag{16}$$

where we have introduced functions [4]

$$R^{(i)} = dK^{(i)}/dr + (1 + \lambda_i)K^{(i)}/r - \beta EK^{(i)}/\lambda_i \tag{17}$$

which satisfy the same equation that $K^{(3)}$

$$\mathrm{T}G = 0, \quad G - \text{any of } K^{(3)}, R^{(1)}, R^{(2)} \tag{18}$$

and the same regularity conditions at the origin. Hence, expressing $K^{(3)}$ via $R^{(i)}$ in virtue of (15),(16) is compatible with (9).

5 On Corben – Schwinger anomaly

When $k_2 = 0, j > 0, g = 2$ our equations (9) with extra condition (15) reduce to those of the Corben – Schwinger theory of the vector particle with gyromagnetic ratio $g = 2$ [4] in Coulomb field. This theory has the well known anomaly, which consists in the absence of normalizable eigenstates of discrete spectrum with $j = 0$ and one of the states with $j = 1$. Consequently, the set of discrete spectrum eigenfunctions is incomplete, what, in its turn, leads to the numerous difficulties, e.g. in scattering theory (see [4] for more details).

But conventional normalization condition for our equation (1) (when $g = 2$) is

$$\int J_0 d^3x = e \tag{19}$$

is associated with the conserved 4-current

$$J_\mu = -ie[B^*_\nu D^\mu B^\nu - B_\nu (D^\mu B^\nu)^*], \tag{20}$$

and it doesn't include terms, proportional to $D_\mu B^\mu$, causing the above-mentioned non-normalizability in Corben – Schwinger theory [4] (in their theory this expression is proportional to the spatial delta-function $\delta(\mathbf{r})$ and integral in (19) becomes divergent for any $j = 0$ states and one of $j = 1$ states).

The key observation, explaining this fact, is following : in our case extra condition (15) is set independently of the equations of motion (9), while its analog in Corben – Schwinger theory was a consequence of the equations of motion and contained in its right hand side not zero but delta-function which lead to the anomaly, as described above.

Thus, we pointed out that our theory hasn't Corben – Schwinger anomaly for $k_2 = 0, j > 0, g = 2$. It can be checked in more straightforward way by calculating the normalization integrals for the eigenfunctions (12) and assuring that they are convergent.

6 Conclusions and discussion

We found eigenvalues and eigenstates of discrete spectrum for the modified Stueckelberg equation in external Coulomb field. We don't give here expressions for the eigenstates of continuous spectrum, since one can easily obtain them in the way similar to the one shown in [12] for the nonrelativistic Coulomb problem.

We gave a hint of the existence of (hidden) parasupersymmetry for the spin-1 states when $k_2 = 0, j > 0, g = 2$. This result naturally generalizes the existence of the supersymmetry for the Dirac equation in Coulomb field [11], and we'll treat it with more details in next papers.

We also pointed out and explained the absence of Corben – Schwinger anomaly in our model. On the other hand, for the case of constant homogeneous magnetic field this model reduces to the one analysed in [7].

Thus, model (1) proved to be good not only in constant homogeneous magnetic field, but in Coulomb field too.

Finally, it would be mentioned that our results may be applied for the construction of the Furry picture at the quantum electrodynamics of the particles described by the equation (1) and in particular for the evaluation of the corresponding Lamb-type shift (cf. [2]) and other similar effects.

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