Multidimensional integrable systems from contact geometry

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Upon having presented a bird's eye view of history of integrable systems, we give a brief review of certain recent advances in the longstanding problem of search for partial differential systems in four independent variables, often referred to as (3+1)-dimensional or 4D systems, that are integrable in the sense of soliton theory. Namely, we review a recent construction for a large new class of (3+1)-dimensional integrable systems with Lax pairs involving contact vector fields. This class contains inter alia two infinite families of such systems, thus establishing that there is significantly more integrable (3+1)-dimensional systems than it was believed for a long time.

In fact, the construction under study yields (3+1)-dimensional integrable generalizations of many well-known dispersionless integrable (2+1)-dimensional systems like the dispersionless KP equation, as well as a first example of a (3+1)-dimensional integrable system with an algebraic, rather than rational, nonisospectral Lax pair.

To demonstrate the versatility of the construction in question, we employ it here to produce novel integrable (3+1)-dimensional generalizations for the following (2+1)-dimensional integrable systems: dispersionless BKP, dispersionless asymmetric Nizhnik–Veselov–Novikov, dispersionless Gardner, and dispersionless modified KP equations, and the generalized Benney system.

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1 Introduction

The theory of integrable systems has its roots in trying to answer a simple and natural question: when an ordinary differential equation, or a system of such equations, can be integrated by quadratures? An important milestone here is the Liouville theorem [30] in classical mechanics giving a sufficient condition for this to occur for an important class of integrable Hamiltonian systems that are, inter alia, of considerable significance for applications, see e.g. [1]; such systems were intensively studied by the researchers from Mathematical Institute of UNAM, see e.g. recent works [2, 3] and references therein.

The next great breakthrough in the field of integrability has occurred about half a century ago with the discovery of the so-called inverse scattering transform (IST) in the seminal work [19]. In [19] it was shown that solving the Cauchy problem for the *nonlinear* Korteweg–de Vries (KdV) equation, a remarkable evolutionary partial differential equation in one dependent and two independent variables satisfied, inter alia, by a certain generating function for intersection numbers of complex curves arising in the Witten conjecture and its proof by Kontsevich, see e.g. [43], can, under certain conditions, be reduced to a sequence of *linear* problems, and the procedure in question became known as IST. This is possible because the KdV equation can be written as a compatibility condition for an overdetermined system of linear equations; such overdetermined systems in the context of integrable systems are called the Lax pairs or the Lax/Lax-type representations, see e.g. [14, 21, 27, 28, 46] for details. An important consequence of the above is the construction of infinitely many explicit exact solutions of the KdV equation – the multisoliton solutions, see e.g. [21, 46] for more details on the latter.

Moreover, existence of a Lax pair for the KdV paved the way to construction [13] of other important classes of explicit exact solutions for this equation, namely, the quasiperiodic finite-gap solutions that are inextricably related to algebraic geometry. Note that making use of the counterparts of these solutions for the KP equation, a natural integrable generalization of the KdV equation to the case of three independent variables, made it possible to prove [39] the longstanding Schottky conjecture in algebraic geometry.

Yet another notable consequence of the presence of the Lax pair is existence of infinitely many nontrivial independent local conservation laws for the KdV equation, see e.g. [21, 28, 31] and references therein, which shows, inter alia, that the associated dynamics is highly regular rather than chaotic.

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It was quickly realized that the KdV equation is by no means an isolated example – there is plenty of partial differential systems that admit 'good' Lax pairs from which infinite hierarchies of conservation laws can be extracted and that are, at least in principle, amenable to the IST. In what follows we shall call *integrable* the partial differential systems with 'good' Lax pairs in the above sense, unlike some authors who use the term integrable also for the systems that can be linearized by an appropriate change of variables like e.g. the Burgers equation.

Note that soliton and multisoliton solutions for KdV and many other integrable systems, as well as other types of exact solutions constructed using the Lax pairs, like e.g. the multi-instanton solutions for the (anti)selfdual Yang–Mills equations obtained using the famous ADHM construction [5], have found significant applications both in physics and in pure mathematics, see for example Donaldson's revolutionary works on geometry of fourdimensional manifolds using instantons, see e.g. [4].

Integrable systems are well known to have a number of remarkable structures attached to them. These include Lax pairs, symmetries, conservation laws, Hamiltonian structures and more, see e.g. [12, 14, 21, 24, 28, 31, 33, 40, 44]. Exploring these structures and their properties can provide one with quite a bit of insight into the behavior of the systems under study and their properties, be these systems integrable or not. For one, the presence of large number of symmetries and/or conservation laws indicates that the system under study has a highly constrained, and hence likely quite regular, dynamics, see e.g. the discussion in [1, 21].

Among integrable partial differential systems, those in four independent variables (often referred to as (3+1)dimensional ones even if all four variables are on equal footing) are of particular interest, as four is the dimension of our spacetime according to general relativity, so gaining a deeper understanding of such systems could be quite significant for possible applications.

Integrable (3+1)-dimensional systems were long believed to be quite scarce, with the known examples being mostly related in some way to the concept of (anti)self-duality, as is the case e.g. for the two most important ones, (anti)self-dual Yang–Mills equations and the (anti)self-dual vacuum Einstein equations; an effective construction for integrable (3+1)-dimensional systems appeared rather elusive.

It was, however, recently shown that this is not the case and there is [36] an effective construction that produces a large new class of integrable (3+1)-dimensional systems. We review this construction in detail, along with a number of new examples, in the next section.

The integrable (3+1)-dimensional systems resulting from the construction in question, as well as the overwhelming majority of previously known ones, are dispersionless in the following sense.

A partial differential system is said to be *dispersionless*, or of *hydrodynamic type*, see e.g. [22, 36, 46, 47] and references therein, if it can be written as a first-order homogeneous quasilinear system, that is,

$$A_1(u)u_{x^1} + A_2(u)u_{x^2} + \dots + A_d(u)u_{x^d} = 0;$$
(1)

here d, M and N are natural numbers, A_i are $M \times N$ matrices, $M \ge N$, $\boldsymbol{u} \equiv (u^1, \dots, u^N)^T$ is the vector of unknown functions and $\vec{x} = (x^1, \dots, x^d)^T$ of independent variables, so $\boldsymbol{u} = \boldsymbol{u}(\vec{x})$; the superscript T here and below indicates the transposed matrix. In what follows all functions will be tacitly assumed sufficiently smooth for all computations to make sense.

Dispersionless systems have many applications in fluid dynamics, which largely is the motivation behind the term hydrodynamic-type systems, and in nonlinear optics and general relativity, see for example [8, 20, 36, 42, 46, 47] and references therein.

Below we shall deal with dispersionless systems in four independent variables (i.e., (3+1)-dimensional ones); the independent variables will be denoted x, y, z, t (thus from now on $\vec{x} = (x, y, z, t)^{T}$), so the systems under study read

$$A_1(\boldsymbol{u})\boldsymbol{u}_x + A_2(\boldsymbol{u})\boldsymbol{u}_y + A_3(\boldsymbol{u})\boldsymbol{u}_z + A_4(\boldsymbol{u})\boldsymbol{u}_t = 0.$$
(2)

In keeping with the above remark that the overwhelming majority of integrable (3+1)-dimensional systems known to date can be written in dispersionless form, let us show that this is the case for the anti-self-dual Yang–Mills equations with a matrix Lie group as a gauge group on \mathbb{R}^4 with metric with neutral signature, the (3+1)-dimensional integrable system of utmost importance for physics.

Indeed, it is known, cf. e.g. [35] and references therein, that upon a proper choice of gauge and of local coordinates on \mathbb{R}^4 these equations boil down to a single equation for the so-called Yang matrix J

$$\partial_z (J^{-1} \partial_x J) + \partial_t (J^{-1} \partial_y J) = 0,$$

that upon introducing another matrix K can be rewritten in dispersionless form as

$$\partial_x J = J \partial_t K, \quad \partial_y J = -J \partial_z K. \tag{3}$$

2 Integrable (3+1)-dimensional systems with contact Lax pairs

For an $h = h(p, \boldsymbol{u})$ define an operator X_h as

$$X_h = h_p \partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z \tag{4}$$

which formally looks exactly like the contact vector field with a contact Hamiltonian h on a contact 3-manifold with local coordinates x, z, p and contact one-form dz + pdx, see [36] for details; cf. also e.g. [1, 9] and references therein for general background on contact geometry.

A linear system of the form [36]

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi) \tag{5}$$

for $\chi = \chi(x, y, z, t, p)$ will be hereinafter referred to as a *(linear) contact Lax pair*. Here p is the so-called variable spectral parameter (recall that $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$, so $\boldsymbol{u}_p \equiv 0$), $f = f(p, \boldsymbol{u})$, $g = g(p, \boldsymbol{u})$ are the *Lax functions* and $L = \partial_y - X_f$ and $M = \partial_t - X_g$ are the *Lax operators*.

The Lax pairs (5) provide a new and natural (3+1)-dimensional generalization of the well-known, see e.g. [36, 47] and references therein, (2+1)-dimensional Lax pairs

$$\chi_y = \mathcal{X}_f(\chi), \quad \chi_t = \mathcal{X}_g(\chi), \tag{6}$$

where $\mathcal{X}_h = h_p \partial_x - h_x \partial_p$, since if $u_z = 0$ and $\chi_z = 0$ then (5) boils down to (6). The class of integrable (2+1)dimensional systems with Lax pairs (6) is known to be quite broad, see e.g. [16, 36, 47] and references therein, so it is natural to ask whether the same holds true for the class of integrable (3+1)-dimensional systems with linear contact Lax pairs (5).

The following result, summarizing the key findings of [36], shows that this is indeed the case and there are infinitely many pairs (f, g) such that the systems for u admitting Lax pairs of the form (5) are genuinely (3+1)-dimensional integrable nonlinear systems transformable into Cauchy–Kowalevski form.

Theorem 1 Contact Lax pairs (5) yield integrable (3+1)-dimensional systems that can be brought into Cauchy-Kowalevski form inter alia for the following two infinite sets of pairs of Lax functions f and g:

$$I. \ f = p^{m+1} + \sum_{i=0}^{m} u_i p^i, \quad g = p^{n+1} + \frac{n}{m} u_m p^n + \sum_{j=0}^{n-1} v_j p^j \text{ with } \mathbf{u} = (u_0, \dots, u_m, v_0, \dots, v_{n-1})^{\mathrm{T}},$$
$$II. \ f = \sum_{i=1}^{m} \frac{a_i}{(p-u_i)}, \quad g = \sum_{j=1}^{n} \frac{b_j}{(p-v_j)} \text{ with } \mathbf{u} = (a_1, \dots, a_m, u_1, \dots, u_m, b_1, \dots, b_n, v_1, \dots, v_n)^{\mathrm{T}},$$

where m and n are arbitrary natural numbers.

Before proceeding to the proof of the above theorem, consider for a moment the Lax equation

$$[\partial_y - X_f, \partial_t - X_g] = 0, \tag{7}$$

expressing the compatibility condition for (5). This compatibility condition can be expressed in a more concise form that simplifies many computations:

Proposition 1 ([36]) The Lax equation (γ) holds iff so does a zero-curvature-type equation

$$f_t - g_y + \{f, g\} = 0, \tag{8}$$

where the bracket $\{,\}$ is the contact bracket in dimension three, namely,

$$\{f,g\} = f_p g_x - g_p f_x - p \left(f_p g_z - g_p f_z\right) + f g_z - g f_z \tag{9}$$

Sketch of proof of Theorem 1. Straightforward but tedious computations [36] show that for Case I the compatibility condition (8) for (5), upon equating to zero the coefficients at all powers of p, yields the system

$$(u_{k})_{t} - (v_{k})_{y} + n (u_{k-n-1})_{z} - m (v_{k-m-1})_{z} + (m+1) (v_{k-m})_{x} - (n+1) (u_{k-n})_{x} + \sum_{i=0}^{m} \left\{ (k-i-1)v_{k-i} (u_{i})_{z} - (i-1)u_{i} (v_{k-i})_{z} - (k+1-i)v_{k+1-i} (u_{i})_{x} + iu_{i} (v_{k+1-i})_{x} \right\} = 0,$$
(10)
$$k = 0, \dots, n+m,$$

where to simplify writing we have adopted a notational convention that $u_{m+1} \stackrel{\text{def}}{=} 1$, $v_{n+1} \stackrel{\text{def}}{=} 1$, $u_i \stackrel{\text{def}}{=} 0$ for i > m+1 and i < 0, $v_j \stackrel{\text{def}}{=} 0$ for j > n+1 and j < 0, and finally $v_n \stackrel{\text{def}}{=} (n/m)u_m$.

It can be shown [36] that (10) is an evolution system in disguise, and hence indeed can be brought into the Cauchy-Kowalevski form: namely, (10) can be solved w.r.t. the derivatives $(u_i)_z$ and $(v_j)_z$ for all i = 0, ..., m and j = 0, ..., n - 1.

Likewise, for Case II we have [36] that the compatibility condition (8) for (5), upon being brought to the common denominator and subsequent equating to zero the coefficients at all powers of p in the numerator, yields the system

$$(u_i)_t + \sum_{j=1}^n \left\{ \left(\frac{b_j}{v_j - u_i} \right)_x - \left(\frac{b_j u_i}{v_j - u_i} \right)_z + \frac{2b_j (u_i)_z}{v_j - u_i} \right\} = 0, \quad i = 1, \dots, m,$$

$$(v_j)_y + \sum_{i=1}^m \left\{ - \left(\frac{a_i}{v_j - u_i} \right)_x + \left(\frac{a_i v_j}{v_j - u_i} \right)_z - \frac{2a_i (v_j)_z}{v_j - u_i} \right\} = 0, \quad j = 1, \dots, n,$$

$$(a_i)_t + \sum_{j=1}^n \left\{ \left(\frac{a_i b_j}{(v_j - u_i)^2} \right)_x + \left(\frac{a_i b_j (v_j - 2u_i)}{(v_j - u_i)^2} \right)_z \right.$$

$$(11) - \frac{3a_i (b_j)_z}{v_j - u_i} + \frac{3a_i b_j (v_j)_z}{(v_j - u_i)^2} \right\} = 0, \quad i = 1, \dots, m,$$

$$(b_j)_y + \sum_{i=1}^m \left\{ \left(\frac{a_i b_j}{(v_j - u_i)^2} \right)_x + \left(\frac{a_i b_j (v_j - 2u_i)}{(v_j - u_i)^2} \right)_z \right.$$

$$(12) - \frac{3a_i (b_j)_z}{v_j - u_i} + \frac{3a_i b_j (v_j)_z}{(v_j - u_i)^2} \right\} = 0, \quad j = 1, \dots, n,$$

that can be brought into Cauchy–Kowalevski form e.g. by passing from t to T = y + t with all other variables intact and then solving w.r.t. T-derivatives for all dependent variables. \Box

Now let us return to the study of general properties of contact Lax pairs (5).

First of all, note that contact Lax pairs (5) belong to a broader class of nonisospectral¹ Lax pairs

$$\chi_{y} = K_{1}(p, \boldsymbol{u})\chi_{x} + K_{2}(p, \boldsymbol{u})\chi_{z} + K_{3}(p, \boldsymbol{u})\chi_{p},$$

$$\chi_{t} = L_{1}(p, \boldsymbol{u})\chi_{x} + L_{2}(p, \boldsymbol{u})\chi_{z} + L_{3}(p, \boldsymbol{u})\chi_{p},$$
(12)

and thus at least in principle are amenable to an appropriate version of the inverse scattering transform, see e.g. [36] and references therein, which paves the way to constructing explicit exact solutions of the nonlinear systems admitting such Lax pairs. For examples of general Lax pairs (12) and discussion thereof see e.g. [11, 15, 26, 32] and references therein; for some examples of dispersive (3+1)-dimensional integrable systems see e.g. [18, 41].

Proposition 2 ([36]) A system (2) admits a contact Lax pair of the form (5) if and only if it admits a nonlinear Lax pair for $\psi = \psi(x, y, z, t)$ of the form

$$\psi_y = \psi_z f(\psi_x/\psi_z, \boldsymbol{u}), \quad \psi_t = \psi_z g(\psi_x/\psi_z, \boldsymbol{u}) \tag{13}$$

with the same functions f and g as in (5).

System (13) is nothing but a pair of nonstationary Hamilton–Jacobi equations of a special form, see e.g. [25, 36] and references therein for nonlinear Lax pairs of this kind. Systems like (13) are a special case of multitime Hamilton–Jacobi systems that were intensively studied in a different context, see e.g. [29].

Let us also point out that using a formal expansion of χ in p enables one, at least in principle, to find an infinite hierarchy of nonlocal conservation laws for the system under study using (5), cf. [36] for details. Now we proceed to illustrate the above general results by several new examples of (3+1)-dimensional integrable systems with linear contact Lax pairs.

¹Roughly speaking, nonisospectrality here refers to the fact that the Lax pairs in question involve the derivatives with respect to p; for nonisospectral Lax pairs in general see e.g. [10, 14, 36, 47] and references therein.

Example 1 Let $u = (u, v, w, r, s)^{\mathrm{T}}$, $f = wp^2 + vp + \alpha u$, and $g = w^2p^3 + kvwp^2 + sp + r$, where k and α are arbitrary constants. For k = 2, $\alpha = 1$ and w = 1 the system in question reduces to the case of m = 1, n = 2 in the first of two classes from Theorem 1, if we identify $u_0 \equiv u, u_1 \equiv v, v_0 \equiv r, v_1 \equiv s$; this case is studied in detail in [36].

The associated nonlinear Lax pair (13) for the above f and g reads

$$\psi_y = \psi_z (w(\psi_x/\psi_z)^2 + v\psi_x/\psi_z + \alpha u), \quad \psi_t = \psi_z (w^2(\psi_x/\psi_z)^3 + kvw(\psi_x/\psi_z)^2 + s\psi_x/\psi_z + r)$$

while the linear contact Lax pair (5) has the form

$$\chi_{y} = (2wp + v)\chi_{x} + (-wp^{2} + \alpha u)\chi_{z} + (w_{z}p^{3} + (v_{z} - w_{x})p^{2} + (\alpha u_{z} - v_{x})p - \alpha u_{x})\chi_{p},$$

$$\chi_{t} = (3w^{2}p^{2} + 2kvwp + s)\chi_{x} + (-2w^{2}p^{3} - kvwp^{2} + r)\chi_{z}$$

$$+ (2ww_{z}p^{4} + (kvw_{z} + kwv_{z} - 2ww_{x})p^{3} + (-kvw_{x} - kwv_{x} + s_{z})p^{2} + (r_{z} - s_{x})p - r_{x})\chi_{p},$$
(14)

and the associated integrable system for u, resulting from the compatibility condition for (14), i.e., from (8) with the above f and g, reads

$$w^{2}(w_{x} - (k-2)v_{z}) = 0,$$

$$\alpha u_{t} - r_{y} - \alpha r u_{z} + \alpha u r_{z} + v r_{x} - \alpha s u_{x} = 0,$$

$$v_{t} - s_{y} - 2\alpha k v w u_{x} - r v_{z} + 2w r_{x} - s v_{x} + v s_{x} + \alpha u s_{z} = 0,$$

$$w(w(2k-3)v_{x} - s_{z} + 2\alpha w u_{z} + k v v_{z} + 2v w_{x} - 2w_{y} + 2\alpha u w_{z}) = 0,$$

$$w_{t} - w r_{z} + 2w s_{x} - 3\alpha w^{2} u_{x} + \alpha k v w u_{z} - k v w v_{x} - k w v_{y}$$

$$+ \alpha k u w v_{z} + (k v^{2} - s) w_{x} - k v w_{y} + (\alpha k u v - r) w_{z} = 0$$
(15)

Let us show that this system is an integrable (3+1)-dimensional generalization for (2+1)-dimensional dispersionless KP, dispersionless modified KP, and dispersionless Gardner equations.

First of all, (15) admits an integrable reduction w = 1, and in Example 2 of [36] it is, up to a slight difference in notation, shown that the system obtained from (15) upon setting k = 2 in addition to w = 1 is an integrable (3+1)-dimensional generalization of the dispersionless KP equation.

Now let us show that (15) for k = 3/2 is a (3+1)-dimensional integrable generalization of the (2+1)-dimensional dispersionless Gardner and dispersionless modified KP equations, see e.g. [17] and references therein for those two. Upon putting $u_z = 0$ in (15) with k = 3/2 we get

$$w^{2}w_{x} = 0,$$

$$\alpha u_{t} - r_{y} + vr_{x} - \alpha su_{x} = 0,$$

$$v_{t} - s_{y} - 3\alpha vwu_{x} + 2wr_{x} - sv_{x} + vs_{x} = 0,$$

$$w(2vw_{x} - 2w_{y}) = 0,$$

$$w_{t} + 2ws_{x} - 3\alpha w^{2}u_{x} - (3/2)vwv_{x} - (3/2)wv_{y} + ((3/2)v^{2} - s)w_{x} - (3/2)vw_{y} = 0$$
(16)

The above system admits a reduction w = 1 and $v = \beta u$, where β is an arbitrary constant, and then we get

$$\alpha u_t - r_y + \beta u r_x - \alpha s u_x = 0,$$

$$\beta u_t - s_y - 3\alpha \beta u u_x + 2r_x + \beta (u s_x - s u_x) = 0,$$

$$2s_x - 3\alpha u_x - (3\beta/2)(\beta u u_x + u_y) = 0$$
(17)

Putting w = 1, k = 3/2, and $v = \beta u$ in the above Lax functions f and g we readily observe that (17) admits a nonlinear Lax pair (where now ψ depends only on x, y, t) of the form

$$\psi_y = \psi_x^2 + \beta u \psi_x + \alpha u, \\ \psi_t = \psi_x^3 + (3/2)\beta u \psi_x^2 + s \psi_x + r,$$

which is nothing but the dispersionless Lax pair for the (2+1)-dimensional dispersionless Gardner equation, see e.g. [17] for the latter, up to a suitable rescaling of dependent and independent variables and minor differences in notation.

Note that (17) reduces to the modified dispersionless KP equation upon putting $\alpha = 0$.

Thus, (15) is an integrable (3+1)-dimensional generalization for the following (2+1)-dimensional integrable systems: dispersionless KP, dispersionless modified KP, and dispersionless Gardner equations.

Example 2 Let $u = (u, v, w, r, s, a, b)^{T}$, $f = p^{3} + vp^{2} + up + w$ and $g = p^{5} + 2vp^{4} + rp^{3} + sp^{2} + ap + b$ that is up to an obvious change of notation the case of m = 3 and n = 5 in class I of Theorem 1.

In analogy with the above, the compatibility condition (8) for the associated contact Lax pair then yields the following nonlinear system:

$$w_{t} - b_{y} - aw_{x} - bw_{z} + ub_{x} + wb_{z} = 0,$$

$$u_{t} - a_{y} - au_{x} + ua_{x} + wa_{z} - bu_{z} + 2vb_{x} - 2sw_{x} = 0,$$

$$3a_{x} - 2b_{z} - r_{y} - va_{z} - 3ru_{x} + 2rw_{z} + ur_{x} + wr_{z} + su_{z} - 2sv_{x} + 2vs_{x} - 8vw_{x} = 0,$$

$$v_{t} - s_{y} + 3b_{x} - av_{x} + 2va_{x} - bv_{z} - vb_{z} - 3rw_{x} - 2su_{x} + sw_{z} + us_{x} + ws_{z} = 0,$$

$$-2r_{z} + 4u_{z} + v_{x} + 4vv_{z} = 0,$$

$$2rv_{z} - vr_{z} + 6vu_{z} - 4vv_{x} + 3r_{x} - 2s_{z} - 5u_{x} + 4w_{z} = 0,$$

$$2ru_{z} - 3rv_{x} + 2vr_{x} + sv_{z} - vs_{z} + 2uv_{x} - 8vu_{x} + 6vw_{z} + 2wv_{z} - 2a_{z} + 3s_{x} - 2v_{y} - 5w_{x} = 0$$
(18)

Upon putting $u_z = 0$ in (18) we can further impose the reduction v = 0, w = 0, s = 0, b = 0, r = 5u/3, after which we end up with the system

$$u_t - a_y - au_x + ua_x = 0, 9a_x - 5u_y - 10uu_x = 0,$$
(19)

which is nothing but a version of the dispersionless BKP equation, cf. for example [17] for the latter, which in turn also is the dispersionless limit of two (2+1)-dimensional integrable equations, namely (2+1)-dimensional versions of Caudrey–Dodd–Gibbon–Sawada–Kotera and Kaup-Kupershmidt equations, cf. for instance [23], Thus (18) is a (3+1)-dimensional integrable generalization of the dispersionless BKP equation.

Example 3 Letting $\boldsymbol{u} = (u, v, w, r, s)^{\mathrm{T}}$ and putting

$$f = u/p + v, \quad g = -(1/3)u^3/p^3 + r/p^2 + uw/p + s,$$
 (20)

yields, upon equating to zero the coefficients at all powers of p in (8), the system

$$u_{t} + uvw_{z} - 2uwv_{z} + vwu_{z} - su_{z} + 2us_{z} - uw_{y} - wu_{y} = 0,$$

$$v_{t} - s_{y} - sv_{z} + vs_{z} = 0,$$

$$r_{y} - 2u^{2}w_{z} - uwv_{x} + 3rv_{z} - vr_{z} + us_{x} = 0,$$

$$2ru_{x} - ur_{x} - (2/3)u^{3}u_{z} - u^{3}v_{x} = 0,$$

$$2ur_{z} + u^{2}u_{y} - u^{2}vu_{z} - 3ru_{z} + 2rv_{x} + (4/3)u^{3}v_{z} - u^{2}w_{x} = 0,$$

(21)

which is a novel integrable (3+1)-dimensional generalization of (2+1)-dimensional dispersionless asymmetric Nizhnik-Veselov-Novikov equation, see Subcase 2.1 in Subsection 4.1 of [17] for the latter and cf. also [45] and references therein; the said (2+1)-dimensional dispersionless system is recovered when we drop the z-dependence by putting $u_z = 0$ (and $\chi_z = 0$) and then imposing the reduction v = s = r = 0.

Example 4 Let $\boldsymbol{u} = (v, w, a_1, \dots, a_n, u_1, \dots, u_n)^{\mathrm{T}}$, when n is any natural number, and

$$f = -p^2/2 - vp - w, \quad g = \sum_{j=1}^n \frac{a_j}{(p - u_j)}$$
 (22)

Then the compatibility condition (8) for the associated contact Lax pair yields the nonlinear system

$$(a_{i})_{t} = (u_{i}^{2}/2 - w)(a_{i})_{z} + 3a_{i}u_{i}v_{z} + u_{i}a_{i}(u_{i})_{z} - (u_{i} + v)(a_{i})_{x} - a_{i}((u_{i})_{x} + v_{x} - 2w_{z}), \quad i = 1, \dots, n,$$

$$(u_{i})_{t} = (u_{i}^{2}/2 - w)(u_{i})_{z} + u_{i}^{2}v_{z} - (u_{i} + v)(u_{i})_{x} + u_{i}(w_{z} - v_{x}) - w_{x}, \quad i = 1, \dots, n$$

$$v_{y} = \frac{1}{2}\sum_{i=1}^{n} (a_{i})_{z},$$

$$w_{y} = \frac{1}{2}\sum_{i=1}^{n} (u_{i}(a_{i})_{z} + a_{i}(u_{i})_{z}) + 2v_{z}\sum_{i=1}^{n} a_{i} - \sum_{i=1}^{n} (a_{i})_{x}$$

$$(23)$$

which is a novel integrable (3+1)-dimensional generalization of generalized Benney system from [47]; we recover the latter upon putting $u_z = 0$ (and $\chi_z = 0$) and subsequently imposing the reduction v = 0. In connection with the above examples let us mention that an integrable (3+1)-dimensional generalization of the (2+1)-dimensional dispersionless Davey–Stewartson equation was recently found, using the contact Lax pair approach from [36], in [34]; see also [7] for some other examples with contact Lax pairs.

We stress that there also exist integrable (3+1)-dimensional systems with finitely many dependent variables that admit contact Lax pairs whose Lax functions are not rational. In particular, below we present a system [38] that, to the best of our knowledge, is the first example of an integrable (3+1)-dimensional system with a nonisospectral Lax pair whose Lax operators are algebraic, rather than rational, in the spectral parameter p.

Theorem 2 ([38]) The seven-component (3+1)-dimensional evolutionary system

$$\begin{split} a_t &= \frac{1}{r^2 - 2rsa + 2s^2b} \bigg((4w(ra - sb) - vr)a_x + ra_y + (2w(2a(ra - sb) - rb) - ur)a_z \\ &+ (vs - 2wr)b_x - sb_y + (2w(sb - ra) + us)b_z + (r - sa)u_x + (ra - 2sb)u_z + (2sb - ra)v_x \\ &+ 2(a(sb - ra) + rb)v_z + 2(a(ar - sb) - rb)w_x + 2(2a^2(ar - sb) - 3bar + 2sb^2)w_z) \bigg), \end{split}$$

$$b_t &= \frac{1}{r^2 - 2rsa + 2s^2b} \bigg(2(2wr - vs)ba_x + 2sba_y + 2(2w(ra - sb) - us)ba_z \\ &+ (2s(va - 2wb) - vr)b_x + (r - 2sa)b_y + (2(usa - wrb) - ur)b_z \\ &+ (2s(b - a^2) + ra)u_x + 2(r - sa)bu_z - 2(r - sa)bv_x - 2(ra - 2sb)bv_z \\ &+ 2(ra - 2sb)bw_x + 4(a(ra - sb) - rb)bw_z) \bigg), \end{split}$$

$$r_t &= \frac{1}{r^2 - 2rsa + 2s^2b} \bigg((vs - 2wr)ra_x - rsa_y - (2w(ra - sb) - us)ra_z + (2wr - vs)sb_x \\ &+ s^2b_y + (wr^2 - us^2)b_z + (sa - r)su_x + (2sb - ra)su_z + (r - sa)rv_x + (ra - 2sb)rv_z \\ &+ (2sb - ra)rw_x - 2(a(ra - sb) - rb)rw_z) \bigg), \end{aligned}$$

$$s_t &= w_x + aw_z + wa_z, \\ u_t &= ar_x + 2br_z - sb_x, \\ v_t &= r_x + ar_z + as_x + 2bs_z - sa_x + sb_z, \end{split}$$

$$w_t = s_x + as_z + sa_z$$

is integrable: it admits a nonisospectral Lax pair of the form (5) with algebraic Lax functions f and g given by

$$f = u + vp + wp^{2} + (r + sp)\sqrt{p^{2} + 2ap + 2b}, \quad g = \sqrt{p^{2} + 2ap + 2b}, \tag{24}$$

namely,

$$\chi_{y} = \frac{1}{g} \bigg(\big(2sp^{2} + (r+3sa+2wg)p + ra + vg + 2sb \big) \chi_{x} + \big(-sp^{3} - (wg+sa)p^{2} + pra + 2rb + ug \big) \chi_{z} \\ + \big(s_{z}p^{4} + (2as_{z} + sa_{z} + r_{z} + gw_{z} - s_{x})p^{3} + ((v_{z} - w_{x})g + 2bs_{z} + ra_{z} + sb_{z} - 2as_{x} - sa_{x} \\ - r_{x} + 2ar_{z})p^{2} + ((u_{z} - v_{x})g + rb_{z} - ra_{x} - sb_{x} - 2bs_{x} - 2r_{x}a + 2br_{z})p - rb_{x} - gu_{x} - 2br_{x} \big) \chi_{p} \bigg),$$

$$\chi_{t} = \frac{1}{g} \bigg((p+a)\chi_{x} + (ap+2b)\chi_{z} + (a_{z}p^{2} + p(b_{z} - a_{x}) - b_{x})\chi_{p} \bigg),$$

and a nonlinear Lax pair of the form (13) with f and g given by (24):

$$\psi_y = u\psi_z + v\psi_x + w\psi_x^2/\psi_z + (r\psi_z + s\psi_x)\sqrt{(\psi_x/\psi_z)^2 + 2a\psi_x/\psi_z + 2b},$$

$$\psi_t = \psi_z\sqrt{(\psi_x/\psi_z)^2 + 2a\psi_x/\psi_z + 2b}.$$

3 Outlook

In the present article we have reviewed a recent construction [36] for a large class of integrable (3+1)-dimensional systems with Lax pairs involving contact vector fields, thus showing inter alia that integrable (3+1)-dimensional systems are by no means as scarce as it was believed for a long time. Let us stress that the class in question contains inter alia infinitely many integrable (3+1)-dimensional systems admitting Lax pairs of the form (5) with the Lax functions f and g rational in p, as well as the first known example [38] of a (3+1)-dimensional integrable system with a nonisospectral Lax pair whose Lax operators are algebraic, rather than rational, in the variable spectral parameter p. Moreover, using the construction in question we have found in Examples 1–4 above four novel integrable (3+1)-dimensional systems which generalize the following (2+1)-dimensional integrable systems: dispersionless modified KP, dispersionless Gardner, dispersionless BKP, and dispersionless asymmetric Nizhnik–Veselov–Novikov equations, and the generalized Benney system.

In closing let us point out two open problems related to the above results

- Can one find an example of integrable system with contact Lax pair (5) such that the associated Lax functions f and g are transcendental, rather than algebraic or rational, in p?
- Is it possible to find examples of integrable (3+1)-dimensional systems with contact Lax pairs (5) such that the systems in question can be brought into Cauchy–Kowalevski form and the vector \boldsymbol{u} of unknown functions has just two or three components?

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